บทความวิจัย

ขอบเขตล่างที่ชัดเจนของพลังงานสถานะพื้นของสสาร ที่ไม่เป็นไปตามหลักการกีดกันใน 2 มิติ

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บทคัดย่อ

ขอบเขตล่างของพลังงานที่สถานะพื้นของสสารที่เป็นกลางประเภทโบซอนใน 2 มิติ ภายใต้อันตรกิริยาคูลอมบ์ โดยที่ประจุบวกถูกกำหนดให้อยู่กับที่ คือ $E_N > -c_B N^2$ ซึ่งได้จากการพิจารณาขอบเขตล่างของพลังงานจลน์ในรูปยกกำลังของอินทิกรัลของ ρ^2 เมื่อ ρ คือความหนาแน่นของอนุภาค เมื่อพิจารณาร่วมกับขอบเขตบนของพลังงานที่สถานะพื้นของสสารประเภทโบซอนใน 2 มิติ $E_N < -0.0002N^2$ ซึ่งนำเสนอโดยมุธาพรและมาโนเคียน (2547) จะได้ค่าขอบเขตของพลังงานที่สถานะพื้นของสสารประเภทโบซอนใน 2 มิติ คือ $-4(1 + Z_{\max})N^2 < E_N < -0.0002N^2$ ในหน่วยริดเบิร์ก ยิ่งไปกว่านั้น จากขอบเขตล่างของพลังงานที่สถานะพื้น $E_N \sim -N^2$ สรุปได้ว่าการยุบตัวของสสารประเภทโบซอนจาก 2 ระบบเป็น 1 ระบบใน 2 มิติ คือความไม่เสถียรอันเนื่องจากการคายพลังงานของระบบที่ประกอบด้วยอนุภาคจำนวนมาก

คำสำคัญ: พลังงานสถานะพื้น ความไม่เสถียร ขอบเขตล่าง สสารประเภทโบซอน

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Rigorous Lower Bounds for the Ground State Energy of Matter without the Exclusion Principle in 2D

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ABSTRACT

The lower bound, $E_N > -c_B N^2$, for the ground state energy in two dimensions of neutral matter of bosonic types with Coulomb interactions with fixed positive charges is derived by considering, in process, lower bound for the kinetic energy as some power of an integral of ρ^2 where ρ is the particle density. Combining with the upper bound in two dimensions, derived by Muthaporn C. and Manoukian E.B. (2004), which is $E_N < -0.0002N^2$, the range of the ground state energy of bosonic matter in two dimensions, which is $-4(1 + Z_{\max})N^2 < E_N < -0.0002N^2$ in Rydberg unit, is possessed. Furthermore, the bound for the ground state energy of bosonic matter $E_N \sim -N^2$ impies that, in two dimensions, the collapse of the two systems into one is unstable as the released energy becomes overwhelming larges for large number of particle.

Keywords: ground state energy, instability, lower bound, bosonic matter

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1. Introduction

There has been much interest in recent years in physics in 2D, e.g. [1, 2, 3, 4], and the role of the spin and statistics theorem. It has thus become important to investigate the nature of matter without the exclusion principle in 2D, "bosonic matter". It is an important theoretical question to investigate if the change of the dimensionality of space will change matter from stable to unstable or explosive phase. To answer such questions, we derive a rigorous lower bound for the ground-state energy E_N of the system with N negatively charged bosons and N motionless, i.e., fixed N positive charges, with Coulombic interactions and show that "bosonic matter" *is unstable* in 2D. We do not, however, dwell upon nature for higher dimensions here, with the exception of some comments made in the concluding section. Some of the present field theories speculate that at early stages of our universe the dimensionality of space was not necessarily coinciding with three, and by a process which may be referred to as compactification of space, the present three-dimensional character of space arose upon the evolution and the cooling down of the universe.

Although, in 2004, Muthaporn and Manoukian [5, 6] obtained an upper bound for the ground-state energy for bosonic matter in 2D, which is $E_N < -0.0002N^2$, the knowledge of lower bound is also important to get an actual estimate range for the ground-state energy and, fortunately, infers its instability . The present paper deals with mathematically rigorous treatment of such system by deriving an explicit lower bound for the ground-state energy E_N without using any trial wave function, we investigate by considering particle density satisfied $\int d^2\,\mathbf{x}\;\rho(\mathbf{x}) = N$ and separate this paper to 5 sections. In section 2, a study of the general lower bound for Coulomb potential is firstly carried out in 2D in order to obtain the lower bounds for Coulomb energy. Secondly, the lower bound for the kinetic energy in 2D in term of $\rho^2(\mathbf{x})$ is derived in section 3. The lower bound for the exact-ground-state energy of matter in 2D is then derived in section 4. Finally, Section 5 deals with our conclusion. Here for completeness, we sketch over derivation of the lower bound by considering the neutral matter composing of kinetic energy and Coulomb potential energy in two dimensions. The Hamiltonian is

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \sum_{i<j}^{N} e^2 \left|\mathbf{x}_i - \mathbf{x}_j\right|^{-1} + \sum_{i<j}^{k} Z_i Z_j e^2 \left|\mathbf{R}_i - \mathbf{R}_j\right|^{-1} - \sum_{i=1}^{N}\sum_{j=1}^{k} Z_j e^2 \left|\mathbf{x}_i - \mathbf{R}_j\right|^{-1}, \qquad (1)$$

where

$$\sum_{i=1}^{k} Z_i = N, \quad k \geq 2, \qquad (2)$$

with fixed positive charges, and $\mathbf{x}_i$, $\mathbf{R}_j$ refer to the position of negative and positive charges, respectively. We note that for $k = 1$, the third term in the right-hand side of (1) will be absent in the expression for H and one would be dealing with an atom. Throughout, we are interested in the case for which $k \neq 1$ relevant to matter.

2. The general bound for Coulomb potential in 2D

Consider a real function $v(\mathbf{x})$ where $\mathbf{x}$ is a vector in 2D, with the properties that the Fourier transform pair is

$$v(\mathbf{x}) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \tilde{v}(\mathbf{k}) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{3}$$

and

$$\tilde{v}(\mathbf{k}) = \int \mathrm{d}^2\mathbf{x}\, v(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{4}$$

such that $v(\mathbf{x}) \geq 0, v(0) < \infty$ and $\tilde{v}(\mathbf{k}) \geq 0$. Let $\phi(\mathbf{x}_j)$ be a real function and

$$\phi(\mathbf{x}_j) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \tilde{\phi}(\mathbf{k}) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}. \tag{5}$$

Let $A_1,\ldots,A_j$ $(j \geq 2)$ be real and positive numbers. We have

$$A_j\, \phi(\mathbf{x}_j) = A_j \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \tilde{\phi}(\mathbf{k})\ \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j} \tag{6}$$

and

$$A_1\, \phi(\mathbf{x}_1) + A_2\, \phi(\mathbf{x}_2) + \cdots + A_k\, \phi(\mathbf{x}_k) = \sum_{j=1}^{k} A_j\, \phi(\mathbf{x}_j). \tag{7}$$

Substituting $\phi(\mathbf{x}_j) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \tilde{\phi}(\mathbf{k}) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}$ into $\sum_{j=1}^{k} A_j \phi(\mathbf{x}_j)$, we obtain

$$\sum_{j=1}^{k} A_j\, \phi(\mathbf{x}_j) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \tilde{\phi}(\mathbf{k}) \left(\sum_{j=1}^{k} A_j\ \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j} \right). \tag{8}$$

Multiplying the integrand on the right-hand side of (8) by $\frac{\sqrt{\tilde{v}(\mathbf{k})}}{\sqrt{\tilde{v}(\mathbf{k})}}$, it follows that

$$\sum_{j=1}^{k} A_j\, \phi(\mathbf{x}_j) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \frac{\tilde{\phi}(\mathbf{k})}{\sqrt{\tilde{v}(\mathbf{k})}} \left(\sum_{j=1}^{k} A_j \sqrt{\tilde{v}(\mathbf{k})} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j} \right). \tag{9}$$

Noting the Cauchy-Schwartz inequality

$$\left(\sum_{j=1}^{n} c_j d_j\right)^2 \le \sum_{j=1}^{n} \left|c_j\right|^2 \sum_{j=1}^{n} \left|d_j\right|^2. \tag{10}$$

Applying (10) to (9), then implies that

$$\left(\sum_{j=1}^{k} A_j\, \phi(\mathbf{x}_j)\right)^2 \le \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \left|\frac{\tilde{\phi}(\mathbf{k})}{\sqrt{\tilde{v}(\mathbf{k})}}\right|^2 \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \left|\sum_{j=1}^{k} A_j \sqrt{\tilde{v}(\mathbf{k})}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}\right|^2. \tag{11}$$

Considering the term $\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \left|\sum_{j=1}^{k} A_j \sqrt{\tilde{v}(\mathbf{k})}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}\right|^2$ on the right-hand side of (11), $v(\mathbf{x})$ can be replace by $v(\mathbf{x}_i - \mathbf{x}_j)$ and $\sum_{j=1}^{k} A_j$ can be replace by $\sum_{i,j=1}^{k} A_i A_j$ we obtain

$$\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \left|\sum_{j=1}^{k} A_j \sqrt{\tilde{v}(\mathbf{k})}\, e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}\right|^2 = \sum_{i,j=1}^{k} A_i A_j v(\mathbf{x}_i - \mathbf{x}_j) \tag{12}$$

where, recalling that $A_1,\dots,A_k$ $(k \ge 2)$ are real and positive numbers and $\tilde{v}(\mathbf{k}) \ge 0$,

$$v(\mathbf{x}_i - \mathbf{x}_j) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\, \tilde{v}(\mathbf{k})\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)}. \tag{13}$$

Substituting (13) to the right-hand side of (11), we obtain

$$\left(\sum_{j=1}^{k} A_j\, \phi(\mathbf{x}_j)\right)^2 \le \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \frac{\left|\tilde{\phi}(\mathbf{k})\right|^2}{\tilde{v}(\mathbf{k})} \sum_{i,j=1}^{k} A_i\, A_j\, v(\mathbf{x}_i - \mathbf{x}_j). \tag{14}$$

Hence (14) can be arranged as

$$\frac{\left(\sum_{j=1}^{k} A_j\, \phi(\mathbf{x}_j)\right)^2}{\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{\left|\tilde{\phi}(\mathbf{k})\right|^2}{\tilde{v}(\mathbf{k})}} \le \sum_{i,j=1}^{k} A_i\, A_j\, v(\mathbf{x}_i - \mathbf{x}_j). \tag{15}$$

Considering $(c - d)^2 \geq 0$, for any real number c,d such that $d > 0$, we have

$$\frac{c^2}{2d} \geq c - \frac{d}{2} \tag{16}$$

Setting

$$c = \sum_{j=1}^{k} A_j \, \phi\left(\mathbf{x}_j\right) \tag{17}$$

and

$$d = \int \frac{\mathrm{d}^2\mathbf{k}}{\left(2\pi\right)^2} \frac{\left|\tilde{\phi}\left(\mathbf{k}\right)\right|^2}{\tilde{v}\left(\mathbf{k}\right)}, \tag{18}$$

then noting the inequality in (16), we infer that

$$\frac{1}{2}\sum_{i,j=1}^{k} A_i \, A_j \, v\left(\mathbf{x}_i - \mathbf{x}_j\right) \geq \sum_{j=1}^{k} A_j \, \phi\left(\mathbf{x}_j\right) - \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{k}}{\left(2\pi\right)^2} \frac{\left|\tilde{\phi}\left(\mathbf{k}\right)\right|^2}{\tilde{v}\left(\mathbf{k}\right)}. \tag{19}$$

Considering the left-hand side of inequality (19), we obtain

$$\sum_{i<j}^{k} A_i \, A_j \, v\left(\mathbf{x}_i - \mathbf{x}_j\right) = \frac{1}{2}\sum_{i,j=1}^{k} A_i \, A_j \, v\left(\mathbf{x}_i - \mathbf{x}_j\right) - \frac{1}{2} v\left(0\right) \sum_{j=1}^{k} A_j^2 \, . \tag{20}$$

Substituting (20) into (19), we obtain

$$\sum_{i<j}^{k} A_i \, A_j \, v\left(\mathbf{x}_i - \mathbf{x}_j\right) \geq \sum_{j=1}^{k} A_j \, \phi\left(\mathbf{x}_j\right) - \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{k}}{\left(2\pi\right)^2} \frac{\left|\tilde{\phi}\left(\mathbf{k}\right)\right|^2}{\tilde{v}\left(\mathbf{k}\right)} - \frac{1}{2} v\left(0\right) \sum_{j=1}^{k} A_j^2 \, . \tag{21}$$

Let $V(\mathbf{x})$ be a real function such that $V(\mathbf{x}) \geq v(\mathbf{x})$ and $\rho(\mathbf{x}_j)$ be also a real function and so far arbitrary then we introduce

$$\phi\left(\mathbf{x}'\right) = \int \mathrm{d}^2\mathbf{x} \, \rho\left(\mathbf{x}\right) V\left(\mathbf{x} - \mathbf{x}'\right). \tag{22}$$

Substituting (22) into (21), and replacing $\mathbf{x}'$ by $\mathbf{x}_j$, we obtain

$$\sum_{i<j}^{k} A_i \, A_j \, v\left(\mathbf{x}_i - \mathbf{x}_j\right) \geq \sum_{j=1}^{k} A_j \int \mathrm{d}^2\mathbf{x} \, \rho\left(\mathbf{x}\right) V\left(\mathbf{x} - \mathbf{x}_j\right) - \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{k}}{\left(2\pi\right)^2} \frac{\left|\tilde{\phi}\left(\mathbf{k}\right)\right|^2}{\tilde{v}\left(\mathbf{k}\right)} - \frac{1}{2} v\left(0\right) \sum_{j=1}^{k} A_j^2 \, . \tag{23}$$

Since $\rho(\mathbf{x})$ and $V(\mathbf{x}-\mathbf{x}')$ are real function, i.e., $\rho(\mathbf{x}) = \rho^*(\mathbf{x})$ and $V(\mathbf{x} - \mathbf{x}') = V^*(\mathbf{x} - \mathbf{x}')$, the Fourier transform of (22) is expressed as

$$\tilde{\phi}(\mathbf{k}) = \int d^2\mathbf{x}' \int d^2\mathbf{x}\, \rho(\mathbf{x}) V^*(\mathbf{x}-\mathbf{x}') e^{-i\mathbf{k}\cdot\mathbf{x}'}. \tag{24}$$

Since $V(\mathbf{x}) = \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \tilde{v}(\mathbf{k}') e^{-i\mathbf{k}'\cdot\mathbf{x}}$, (24) can be written as

$$\tilde{\phi}(\mathbf{k}) = \int d^2\mathbf{x}\, \rho(\mathbf{x}) \int d^2\mathbf{k}' \tilde{V}^*(\mathbf{k}') e^{-i\mathbf{k}'\cdot\mathbf{x}} \int \frac{d^2\mathbf{x}'}{(2\pi)^2} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}'}. \tag{25}$$

Applying an integral representation of the delta function in 2D to (25), we obtain

$$\tilde{\phi}(\mathbf{k}) = \tilde{\rho}(\mathbf{k}) \tilde{V}^*(\mathbf{k}). \tag{26}$$

In the same way, $\tilde{\phi}^*(\mathbf{k})$ can be written as

$$\tilde{\phi}^*(\mathbf{k}) = \tilde{\rho}^*(\mathbf{k}) \tilde{V}(\mathbf{k}) \tag{27}$$

Since $|\tilde{\phi}(\mathbf{k})|^2 = \tilde{\phi}^*(\mathbf{k})\tilde{\phi}(\mathbf{k})$, noting (26) and (27), we obtain

$$|\tilde{\phi}(\mathbf{k})|^2 = |\tilde{\rho}(\mathbf{k})|^2 |\tilde{V}(\mathbf{k})|^2. \tag{28}$$

Hence, by using (28), we have

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{|\tilde{\phi}(\mathbf{k})|^2}{\tilde{v}(\mathbf{k})} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} |\tilde{\rho}(\mathbf{k})|^2 \frac{|\tilde{V}(\mathbf{k})|^2}{\tilde{v}(\mathbf{k})} \tag{29}$$

Since $V(\mathbf{y}) \ge v(\mathbf{y})$, $V(\mathbf{y}-\mathbf{y}')$ in the right-hand side of (24) can be replaced by $v(\mathbf{y}-\mathbf{y}')$. In analogy to $V(\mathbf{x})$ which $V(\mathbf{y}) \ge v(\mathbf{y})$, we may introduce

$$\tilde{\varphi}^*(\mathbf{k}) = \int d^2\mathbf{y}\, \rho(\mathbf{y}) \tilde{v}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{y}}. \tag{30}$$

Dividing (30) by $\tilde{v}(\mathbf{k})$ on both sides, we obtain

$$\frac{\tilde{\varphi}^*(\mathbf{k})}{\tilde{v}(\mathbf{k})} = \int d^2\mathbf{y}\, \rho(\mathbf{y}) e^{i\mathbf{k}\cdot\mathbf{y}}. \tag{31}$$

Multiply (31) by $\tilde{\phi}(\mathbf{k})$, also noting (24), it follows that

$$\frac{\tilde{\phi}(\mathbf{k})\tilde{\varphi}^*(\mathbf{k})}{\tilde{v}(\mathbf{k})} = \int \mathrm{d}^2\mathbf{x}' \int \mathrm{d}^2\mathbf{x}\, \rho(\mathbf{x}) V(\mathbf{x}-\mathbf{x}') \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'} \int \mathrm{d}^2\mathbf{y}\, \rho(\mathbf{y}) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{y}}. \tag{32}$$

Dividing (32) by $(2\pi)^2$ then integrating with respects to $\mathbf{k} \in \mathbb{R}^2$ $\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}$, we have

$$\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \frac{\tilde{\phi}(\mathbf{k})\tilde{\varphi}^*(\mathbf{k})}{\tilde{v}(\mathbf{k})} = \int \mathrm{d}^2\mathbf{x}' \int \mathrm{d}^2\mathbf{x}\, \rho(\mathbf{x}) V(\mathbf{x}-\mathbf{x}') \rho(\mathbf{x}'). \tag{33}$$

We rewrite (33) as

$$\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \frac{\tilde{\phi}(\mathbf{k})\tilde{\varphi}^*(\mathbf{k})}{\tilde{v}(\mathbf{k})} = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \left|\tilde{\rho}(\mathbf{k})\right|^2 \tilde{V}(\mathbf{k}) \tag{34}$$

Hence, from (33) and (34), we get

$$\int \mathrm{d}^2\mathbf{x}' \int \mathrm{d}^2\mathbf{x}\, \rho(\mathbf{x}) V(\mathbf{x}-\mathbf{x}') \rho(\mathbf{x}') = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \left|\tilde{\rho}(\mathbf{k})\right|^2 \tilde{V}(\mathbf{k}). \tag{35}$$

Substituting (29) and (35) into (23), we then have

$$\begin{aligned} \sum_{i<j}^{k} A_i A_j v(\mathbf{x}_i - \mathbf{x}_j) \geq \sum_{j=1}^{k} A_j \int \mathrm{d}^2\mathbf{x}\, \rho(\mathbf{x}) V(\mathbf{x}-\mathbf{x}_j) - \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \left|\tilde{\rho}(\mathbf{k})\right|^2 \frac{\left|\tilde{V}(\mathbf{k})\right|^2}{\tilde{v}(\mathbf{k})} + \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \left|\tilde{\rho}(\mathbf{k})\right|^2 \tilde{V}(\mathbf{k}) \\ -\frac{1}{2}\int \mathrm{d}^2\mathbf{x}' \int \mathrm{d}^2\mathbf{x}\, \rho(\mathbf{x}) V(\mathbf{x}-\mathbf{x}') \rho(\mathbf{x}') - \frac{1}{2} v(0) \sum_{j=1}^{k} A_j^2. \end{aligned} \tag{36}$$

Since $V(\mathbf{x}) \geq v(\mathbf{x}) \geq 0$, hence $\sum_{i<j}^{k} A_i A_j V(\mathbf{x}_i - \mathbf{x}_j) \geq \sum_{i<j}^{k} A_i A_j v(\mathbf{x}_i - \mathbf{x}_j)$. Then substituting (35) into (36), we obtain

$$\begin{aligned} \sum_{i<j}^{k} A_i A_j V(\mathbf{x}_i - \mathbf{x}_j) \geq \sum_{j=1}^{k} A_j \int \mathrm{d}^2\mathbf{x}\, \rho(\mathbf{x}) V(\mathbf{x}-\mathbf{x}_j) - \frac{1}{2}\int \mathrm{d}^2\mathbf{x}' \int \mathrm{d}^2\mathbf{x}\, \rho(\mathbf{x}) V(\mathbf{x}-\mathbf{x}') \rho(\mathbf{x}') \\ -\frac{1}{2} v(0) \sum_{j=1}^{k} A_j^2 - \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \left|\tilde{\rho}(\mathbf{k})\right|^2 \left[\frac{\left|\tilde{V}(\mathbf{k})\right|^2}{\tilde{v}(\mathbf{k})} - \tilde{V}(\mathbf{k})\right] \end{aligned} \tag{37}$$

where, needless to say. $\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \left|\tilde{\rho}(\mathbf{k})\right|^2 \tilde{V}(\mathbf{k})$ is real. Let $v(\mathbf{x}) = e^2(1-\mathrm{e}^{-\lambda|\mathbf{x}|})/|\mathbf{x}|$ with $\lambda > 0$. The Fourier transform of $v(\mathbf{x})$ is

$$\tilde{v}(\mathbf{k}) = 2\pi e^2 \left(\frac{\sqrt{\lambda^2 + k^2} - k}{k\sqrt{\lambda^2 + k^2}} \right) \tag{38}$$

We now introduce the Yukawa potential.

$$V_\lambda(\mathbf{x}) = \frac{e^2 e^{-\lambda|\mathbf{x}|}}{|\mathbf{x}|} \quad , \lambda > 0 \tag{39}$$

and evaluate the Fourier transform. Letting $\lambda \longrightarrow 0$ we recover the Coulomb potential from (39). It was, in fact, in response to the short range of nuclear forces that Yukawa introduced λ. For electromagnetism where the range is infinite, λ becomes zero and thenfore $V_\lambda(\mathbf{x})$ reduces to the Coulomb potential i.e. $V_{\lambda\to 0}(\mathbf{x}) = \frac{e^2}{|\mathbf{x}|}$. Thus, on noting (39), the Fourier transform of the Coulomb potential in 2D is

$$\tilde{V}_{\lambda\to 0}(\mathbf{k}) = \lim_{\lambda\to 0} \frac{2\pi e^2}{\sqrt{k^2 + \lambda^2}} = \frac{2\pi e^2}{k}. \tag{40}$$

For $v(0)$, where $v(\mathbf{x}) = e^2(1 - \mathrm{e}^{-\lambda|\mathbf{x}|})/|\mathbf{x}|$ we define $v(0)$ by using Taylor series, as

$$v(0) = \lim_{|\mathbf{x}|\to 0} v(\mathbf{x}) = e^2\lambda. \tag{41}$$

Substituting (38), (40) and (41) into (37), we obtain the general Coulomb potential bound ($k \geq 2$) as

$$\sum_{i<j}^{k} \frac{e^2 A_i A_j}{|\mathbf{x}_i - \mathbf{x}_j|} \geq \sum_{j=1}^{k} e^2 A_j \int d^2\mathbf{x} \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_j|} - \frac{e^2}{2} \int d^2\mathbf{x}' \int d^2\mathbf{x} \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \frac{e^2\lambda}{2} \sum_{j=1}^{k} A_j^2 - \pi e^2 \int d^2\mathbf{x}\, \rho^2(\mathbf{x}) \left(\frac{1}{\left(\sqrt{\lambda^2 + k^2} - k\right)} \right). \tag{42}$$

Noting Minkowski inequality, we have $-\left(\frac{1}{\sqrt{\lambda^2 + k^2} - k} \right) \leq -\left(\frac{1}{\lambda} \right)$. Then, applying to the forth term of the right-hand side of (42), it follows that

$$\begin{aligned} &\sum_{j=1}^{k} e^2 A_j \int d^2\mathbf{x} \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_j|} - \frac{e^2}{2} \int d^2\mathbf{x}' \int d^2\mathbf{x} \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \frac{e^2\lambda}{2} \sum_{j=1}^{k} A_j^2 - \pi e^2 \int d^2\mathbf{x}\, \rho^2(\mathbf{x}) \left(\frac{1}{\left(\sqrt{\lambda^2 + k^2} - k\right)} \right) \\ &\leq \sum_{j=1}^{k} e^2 A_j \int d^2\mathbf{x} \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_j|} - \frac{e^2}{2} \int d^2\mathbf{x}' \int d^2\mathbf{x} \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \frac{e^2\lambda}{2} \sum_{j=1}^{k} A_j^2 - \frac{\pi e^2}{\lambda} \int d^2\mathbf{x}\, \rho^2(\mathbf{x}). \end{aligned} \tag{43}$$

From (42) and (43), in order to obtain the general Coulomb potential in 2D, we can consider only the case that $\lambda = \lambda_0$ which

$$\sum_{i<j}^{k}\frac{e^2A_i\,A_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|}=\sum_{j=1}^{k}e^2A_j\int d^2\mathbf{x}\frac{\rho(\mathbf{x})}{\left|\mathbf{x}-\mathbf{x}_j\right|}-\frac{e^2}{2}\int d^2\mathbf{x}'\int d^2\mathbf{x}\frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}-\frac{e^2\lambda_0}{2}\sum_{j=1}^{k}A_j^2-\frac{\pi e^2}{\lambda_0}\int d^2\mathbf{x}\,\rho^2(\mathbf{x}). \tag{44}$$

In the Hamiltonian, it is then straightforward to apply (44) twice, once to the repulsive potentials, the second term in the right-hand side of (1). Let A_i, A_j = 1 and $k \longrightarrow N$, we obtain

$$\sum_{i<j}^{N}\frac{e^2}{\left|\mathbf{x}_i-\mathbf{x}_j\right|}=\sum_{j=1}^{N}e^2\int d^2\mathbf{x}\frac{\rho(\mathbf{x})}{\left|\mathbf{x}-\mathbf{x}_j\right|}-\frac{e^2}{2}\int d^2\mathbf{x}'\int d^2\mathbf{x}\frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}-\frac{e^2\lambda_0}{2}\sum_{j=1}^{N}(1)-\frac{\pi e^2}{\lambda_0}\int d^2\mathbf{x}\rho^2(\mathbf{x}) \tag{45}$$

and again, to the repulsive potentials, the third term in the right-hand side of (1). Let $A_i = Z_i$, $A_j = Z_j$ and $\mathbf{x}_j \longrightarrow \mathrm{R}_j$ for $k \geq 2$, we also obtain

$$\sum_{i<j}^{k}\frac{e^2Z_iZ_j}{\left|\mathbf{R}_i-\mathbf{R}_j\right|}=\sum_{j=1}^{k}e^2Z_j\int d^2\mathbf{x}\frac{\rho(\mathbf{x})}{\left|\mathbf{x}-\mathbf{R}_j\right|}-\frac{e^2}{2}\int d^2\mathbf{x}'\int d^2\mathbf{x}\,\frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}-\frac{e^2\lambda_0}{2}\sum_{j=1}^{k}Z_j^2-\frac{\pi e^2}{\lambda_0}\int d^2\mathbf{x}\,\rho^2(\mathbf{x}). \tag{46}$$

We substitute $\sum_{i=1}^{k} Z_i = N$ where $k \geq 2$, $\sum_{i=1}^{k}(1) = N$, (45) and (46) into (1), the ground state energy, $\langle\psi|\,H\,|\psi\rangle$ with $k \geq 2$, is expressed as

$$\langle\psi|H|\psi\rangle=T+\left\langle\psi\left|\sum_{j=1}^{N}e^2\int d^2\mathbf{x}\frac{\rho(\mathbf{x})}{\left|\mathbf{x}-\mathbf{x}_j\right|}\right|\psi\right\rangle+\left\langle\psi\left|\sum_{j=1}^{k}e^2Z_j\int d^2\mathbf{x}\frac{\rho(\mathbf{x})}{\left|\mathbf{x}-\mathbf{R}_j\right|}\right|\psi\right\rangle-\left\langle\psi\left|\frac{2\pi e^2}{\lambda_0}\int d^2\mathbf{x}\,\rho^2(\mathbf{x})\right|\psi\right\rangle$$

$$-\left\langle\psi\left|\frac{e^2\lambda_0}{2}\left(N+\sum_{i=1}^{k}Z_i^2\right)\right|\psi\right\rangle-\left\langle\psi\left|e^2\int d^2\mathbf{x}'\int d^2\mathbf{x}\,\frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}\right|\psi\right\rangle-\left\langle\psi\left|\sum_{i=1}^{N}\sum_{j=1}^{k}\frac{e^2Z_j}{\left|\mathbf{x}_i-\mathbf{R}_j\right|}\right|\psi\right\rangle \tag{47}$$

Where

$$T=\left\langle\psi\left|\sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m}\right|\psi\right\rangle. \tag{48}$$

3. The lower bound for the kinetic energy in 2D

For the case of bosonic (of spin 0 for simplicity), in multi-particle systems, for example, the particle density is written as

$$\rho(\mathbf{x}) = N\int \mathrm{d}^2\mathbf{x}_2 \ldots \mathrm{d}^2\mathbf{x}_N \left|\psi\left(\mathbf{x},\mathbf{x}_2,\ldots,\mathbf{x}_N\right)\right|^2 \tag{49}$$

where ψ is an N -boson symmetric normalized wavefunction.

Noting (47) and (49), for $k = 1$, we obtain the ground-state energy of N identical bosons as

$$\langle\psi|H|\psi\rangle = T - 2e^2\left(\frac{\pi N}{2}\right)^{1/2}\left(\int \mathrm{d}^2\mathbf{x}\,\rho^2(\mathbf{x})\right)^{1/2} - Ne^2\int \mathrm{d}^2\mathbf{x}\frac{\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{R}|}. \tag{50}$$

And, on noting (47) to (49), for $k \geq 2$, we obtain the ground-state energy of N identical bosons as

$$\langle\psi|H|\psi\rangle = T - \left\langle\psi\left|\frac{2\pi e^2}{\lambda_0}\int \mathrm{d}^2\mathbf{x}\,\rho^2(\mathbf{x})\right|\psi\right\rangle - \left\langle\psi\left|\frac{e^2\lambda_0}{2}\left(N+\sum_{i=1}^{k}Z_i^2\right)\right|\psi\right\rangle. \tag{51}$$

Optimizing (51) over λ_0 gives

$$\lambda_0 = \left(4\pi\frac{\int \mathrm{d}^2\mathbf{x}\,\rho^2(\mathbf{x})}{\left(N+\sum_{i=1}^{k}Z_i^2\right)}\right)^{1/2}. \tag{52}$$

Substituting (52) into (51) gives the remarkably simple bound, for $k \geq 2$, as

$$\langle\psi|H|\psi\rangle = T - 2e^2\pi^{1/2}\left(N+\sum_{i=1}^{k}Z_i^2\right)^{1/2}\left(\int \mathrm{d}^2\mathbf{x}\,\rho^2(\mathbf{x})\right)^{1/2}. \tag{53}$$

This suggests to use a lower bound for T which is some power of an integral of $\rho^2(\mathbf{x})$.

To the above end, we may apply the Schwinger inequality [7] for the number of eigenvalues (counting degeneracy) $\leq -\xi$, $\xi > 0$, (if any) of a Hamiltonian $\frac{\mathbf{p}^2}{2m} - f(\mathbf{x})$, to (53) for $k \geq 2$ in 2 dimensions to obtain the following inequality

$$N_{-\xi}\left(H_0 - f(\mathbf{x})\right) \le \left(\frac{m}{2\pi\hbar^2\xi}\right)\left(\int d^2\mathbf{x}\left[f(\mathbf{x})\right]^2\right) \tag{54}$$

where $f(\mathbf{x}) \ge 0$. Recently, an exact functional expression for $N_{-\xi}\,(H_0 - f(\mathbf{x}))$ derived by Manoukian and Limboonsong [8] is not an upper bound as in (54).

For $N_{-\xi}\left(\frac{\mathbf{p}^2}{2m} - f(\mathbf{x})\right) < 1,$ and any $\delta > 0$ we may choose ξ in (54) such that

$$-\xi = -\left(\frac{m}{2\pi\hbar^2}\right)(1+\delta)\left(\int d^2\mathbf{x}\left[f(\mathbf{x})\right]^2\right) \tag{55}$$

so that $N_{-\xi}\left(\frac{\mathbf{p}^2}{2m} - f(\mathbf{x})\right) < 1,$ which implies that $N_{-\xi}\left(\frac{\mathbf{p}^2}{2m} - f(\mathbf{x})\right) = 0$, The right-hand side of (55) provides a lower bound to the spectrum of $\left(\frac{\mathbf{p}^2}{2m} - f(\mathbf{x})\right)$ since its spectrum would then be emply for energies $\le -\xi$. Therefore, (55) gives the following lower bound for the ground-state energy of the Hamiltonian which is

$$-\left(\frac{m}{2\pi\hbar^2}\right)(1+\delta)\left(\int d^2\mathbf{x}\left[f(\mathbf{x})\right]^2\right). \tag{56}$$

To obtain the lower bound of T in one particle systems, we first consider one particle which $\int d^2\mathbf{x}\,\rho(\mathbf{x}) = 1$ and define positive function $f(\mathbf{x}) = \gamma\frac{\rho^{\alpha}(\mathbf{x})}{\int d^2\mathbf{x}\,\rho^{\alpha+1}(\mathbf{x})}T$ where $\gamma,\alpha > 0$ and $f(\mathbf{x})$ is not the potential energy for any Hamiltonian. This is introduced in order to be able to obtain a lower bound for T. Accordingly, with $\gamma = 2$ and $\alpha = 1$, we obtain the positive function $f(\mathbf{x})$ in term of $\rho^2(\mathbf{x})$ as

$$f(\mathbf{x}) = 2\frac{\rho(\mathbf{x})}{\int d^2\mathbf{x}\,\rho^2(\mathbf{x})}T. \tag{57}$$

For N identical bosons, by using (49) and (57) for each i^{th} particle, where $T_i = \left\langle\psi\left|\sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m}\right|\psi\right\rangle$ we obtain

$$\left\langle \psi \left| \sum_{i=1}^{N} \left[\frac{\mathbf{p}_i^2}{2m} - f\left(\mathbf{x}_i\right) \right] \right| \psi \right\rangle = -T \tag{58}$$

Now, in order to obtain a lower bound to the lower of the spectrum of the "Hamiltonian" in (58), we can put N bosons in the same state without Pauli's exclusion principle (put all of the N bosons at the bottom of the spectrum of $\left(\frac{\mathbf{p}^2}{2m} - f(\mathbf{x})\right)$). Hence the Hamiltonian (58) is bounded below by N times the ground-state energy in (56). We have for N identical bosons,

$$\left\langle \psi \left| \sum_{i=1}^{N} \left[\frac{\mathbf{p}_i^2}{2m} - f\left(\mathbf{x}_i\right) \right] \right| \psi \right\rangle \geq -N\xi. \tag{59}$$

Substituting (55) into (59) then compare with (58), we finally have the expectation value of the kinetic energy T, for N identical bosons, as

$$T \geq \frac{1}{N}\left(\frac{\pi\hbar^2}{2m}\right)\frac{1}{(1+\delta)}\left(\int \mathrm{d}^2\mathbf{x}\, \rho^2\left(\mathbf{x}\right)\right) \tag{60}$$

for any $\delta > 0$.

4. Lower bound for the ground state energy of bosonic matter in 2D

Substituting (60) into (53), the lower bound for the ground state energy of bosonic matter in 2D is expressed as

$$\langle \psi | H | \psi \rangle \geq \frac{1}{N}\left(\frac{\pi\hbar^2}{2m}\right)\frac{1}{(1+\delta)}\left(\int \mathrm{d}^2\mathbf{x}\, \rho^2\left(\mathbf{x}\right)\right) - 2e^2\pi^{1/2}\left(N + \sum_{i=1}^{k} Z_i^2\right)^{1/2}\left(\int \mathrm{d}^2\mathbf{x}\, \rho^2\left(\mathbf{x}\right)\right)^{1/2}. \tag{61}$$

Upon setting $\left(\int \mathrm{d}^2\mathbf{x}\, \rho^2(\mathbf{x})\right)^{1/2} = A, \left(\frac{\pi\hbar^2}{2m}\right)\frac{1}{(1+\delta)} = b,$ (61) becomes, for (k 2)

$$\begin{aligned}
\langle \psi | H | \psi \rangle &\geq \frac{b}{N} A^2 - 2e^2\pi^{1/2}\left(N + \sum_{i=1}^{k} Z_i^2\right)^{1/2} A \\
&= \frac{b}{N}\left(A - e^2\pi^{1/2}\frac{N}{b}\left(N + \sum_{i=1}^{k} Z_i^2\right)^{1/2}\right)^2 - \frac{e^4\pi N}{b}\left(N + \sum_{i=1}^{k} Z_i^2\right)
\end{aligned}$$

$$> -\frac{e^4 \pi N}{b}\left(N + \sum_{i=1}^{k} Z_i^2\right)$$

$$\langle \psi | H | \psi \rangle > -4\left(\frac{me^4}{2\hbar^2}\right) N^2 \left(1 + \frac{\sum_{i=1}^{k} Z_i^2}{N}\right) \tag{62}$$

where δ has taken arbitrarily small, $\delta \ll N$.

Finally, using the bound

$$\sum_{i=1}^{k} Z_i^2 \leq Z_{\text{MAX}} \sum_{i=1}^{k} Z_i = Z_{\text{MAX}} N \tag{63}$$

where $Z_{\max}$ corresponds to the nucleus with largest charge in units of $|e|$.

Substituting (63) into (62), we obtain the lower bound for the ground state energy of bosonic matter in 2D as

$$\langle \psi | H | \psi \rangle > -c_B N^2, \qquad c_B = 4\left(1 + Z_{\text{MAX}}\right)\left(\frac{me^4}{2\hbar^2}\right) \tag{64}$$

5. Conclusion

The lower bound for the ground state energy of bosonic matter depends on the particle number square, N^2. It is interesting to note that even if $Z_1 = ... = Z_N = 1$ in (64), the coefficient of N^2 is of the order 8. For $1 \leq Z_i \leq Z_{\max}$ the maximum coefficient of N^2 is $4(1+Z_{\max})$. When we combine the result in (64) with the upper bound, deriving by Muthaporn, C. and Manoukian, E. B. [6], we get the range of the ground state energy of bosonic matter of N particles as $-4(1+Z_{\max})\, N^2 < E_N < -0.0002\, N^2$ in unit of $\left(\frac{me^4}{2\hbar^2}\right)$.

For further discussion on bosonic matter (matter without the exclusion principle) behavior in two dimensions, we obtain an N^2 behavior which is to be compared to the $N^{5/3}$ one in three dimensions [9-14], which is implying evenmore violent collapse of such system in two dimensions. The ground-state energy E_N in two dimensions forms $-E_N - N^{\alpha}$ with $\alpha > 1$ as same as [9, 10, 11, 15] in three dimensions. Also such a power law behavior, $\alpha > 1$, in which, for two dimensions, $\alpha = 2$ implies instability as the formation of a single system consisting of $(N + N)$

particles favored over two separate systems brought together each consisting of N particles, and the energy released upon the collapse of the two systems into one, being proportional to $[(2N)^2 - 2(N)^2]$, will be overwhelmingly large for realistic large N, e.g., $N \sim 10^{23}$. Regarding to such a collapse Dyson states [9, 10]: "[Bosonic] matter in bulk would collapse into a condensed high-density phase. The assembly of any two macroscopic objects would release energy comparable to that of an atomic bomb. Matter without the exclusion principle, bosonic matter, is unstable in two dimensions".

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