บทความวิจัย

วิธีการเบื้องต้นในการหาจุดตรึงของฟังก์ชันต่อเนื่อง

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บทคัดย่อ

การมีอยู่ของจุดตรึงมีความสำคัญต่อคณิตศาสตร์ และ วิทยาศาสตร์ ในหลายๆ ด้าน จากผลการศึกษาที่ผ่านมา เป็นที่ทราบกันดีว่า ถ้าฟังก์ชัน $f:\mathbb{R}\to\mathbb{R}$ เป็นฟังก์ชันการหดตัว แล้ว f มีจุดตรึงเพียงจุดเดียวเท่านั้น อย่างไรก็ตาม ฟังก์ชันพื้นฐานส่วนใหญ่ เช่น ฟังก์ชันพหุนาม, ฟังก์ชันเลขชี้กำลัง และฟังก์ชันลอการิทึมไม่สอดคล้องกับสมการการหดตัว ในบทความนี้เราเสนอแนวคิดเบื้องต้นและทำการพิสูจน์ในเชิงวิเคราะห์เพื่อการตรวจสอบถึงการมีอยู่ของจุดตรึงของฟังก์ชันค่าจริงที่ต่อเนื่องซึ่งสอดคล้องเงื่อนไขชุดหนึ่ง โดยการวิเคราะห์ความชันของฟังก์ชันจะสามารถยืนยันถึงการมีอยู่ของจุดตรึงของฟังก์ชัน ผลที่ตามมาและตัวอย่างที่เกี่ยวข้องได้ถูกนำเสนอในบทความนี้ด้วย

คำสำคัญ: การส่งแบบหดตัว ฟังก์ชันต่อเนื่อง จุดตรึง

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An Elementary Method to Determine a Fixed Point of a Continuous Function

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ABSTRACT

The existence of a fixed point is important in several areas of mathematics and other sciences. From a classical result it is well known that if a function $f:\mathbb{R}\rightarrow\mathbb{R}$ is a contraction function then f has a unique fixed point. However, most elementary function such as polynomial, exponential and logarithm do not satisfy the contraction property. In this paper, we give some elementary ideas and provide simple analytical proofs to determine the existence of a fixed point of a real valued continuous function that satisfies the certain conditions. In fact the slope of functions will be analyzed to determine when a fixed point exists. Several consequences and related examples are also given.

Keywords: Contraction mapping, Continuous function, Fixed point

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Introduction

In calculus, continuity, differentiability and integrability of functions are commonly the subjects that we are interested in. However, there is another property of a function called "fixed point" which is also an important property in pure and applied. In many mathematical problems, the existence of a fixed point determines the existence of the solutions of the problems. From [1] we have known that if a function $f:\mathbb{R}\to\mathbb{R}$ satisfies a property called contraction, that is $|f(x)-f(y)|\le k|x-y|$ for all $x, y \in \mathbb{R}$ and for fixed $k \in (0,1)$, then f has a uniquefixed point. In [2] Beardon weaken the inequality and show that the existence of a fixed point is still true in the case of the extended real number. Unfortunately, most elementary functions such as polynomial, exponential and logarithm do not satisfy the contraction property. The aim of this article is to give elementary and basic ideas to determine when a continuous function has a fixed point and when it does not. In fact, we provide some elementary proofs to determine the existence of a fixed point of a continuous function by analyzing the slope of the function. More precisely, we will show that for $a \in \mathbb{R}$ and $k \in (0,1)$ if $f:[a,\infty)\to\mathbb{R}$ is continuous and satisfies

$$f(x)-f(y)\le k(x-y) \text{ for all } x,y\in[a,\infty) \tag{1}$$

then f has a unique fixed point if and only if $f(a)\ge a$. Consequently, we obtain several interesting corollaries to guarantee the existence of a fixed point.

Definitions and Backgrounds

This section will provide the definition of a fixed point and some other related materials and examples.

Definition 1. Let $f:\mathbb{R}\to\mathbb{R}$ be a real valued function. We say $c\in\mathbb{R}$ is a *fixed point* of the function f if $f(c)=c$.

Example 1. $f(x)=x^3-3x$. Then it can be easy to see that -2, 0 and 2 are all fixed points of f as shown in the figure 1.

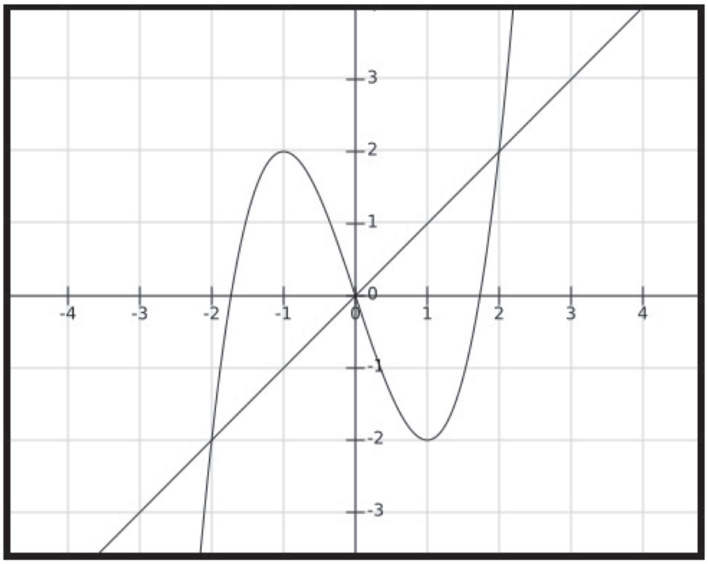

Figure 1

Definition 2. Let $0 < k < 1$. A *contraction* of a real valued function is a function $f:\mathbb{R}\rightarrow\mathbb{R}$ satisfying

$$|f(x)-f(y)|\leq k|x-y| \text{ for all } x,y\in\mathbb{R}. \qquad (2)$$

We have known from [1] that if $f:\mathbb{R}\rightarrow\mathbb{R}$ satisfies (2) then f has a unique fixed point. It is not hard to see that the function in the example 1 does not satisfy with this property. In fact, most elementary functions such as $2x$, e^{-x} and $\log x + 1$ which have a fixed point do not satisfy the contraction property either. Therefore the following section and its consequences are developed.

Results

Proposition 1. *Let* $a \in \mathbb{R}$ and $k < 1$. *Suppose that* $f :[a, \infty) \rightarrow \mathbb{R}$ *is a continuous function satisfying*

$$f(x)-f(y) \leq k(x-y) \text{ for all } x, y \in[a, \infty).$$

Then either f *has a unique fixed point or* $f(x) < x$ *for all* $x \in[a, \infty)$.

Proof. Case 1. $f(a) \geq a$

Fist, we will show that f has a fixed point. To do this, we will find $b > a$ and show that $f(b) \leq b$. Since $f(a) \geq a$ and $k < 1$, by Archimedean Property, we have $f(a)-ka \leq (1-k)b$ for some $b > a$. Thus $f(b)=f(b)-f(a) + f(a) \leq k(b-a) + f(a) = kb + f(a) - ka \leq kb + (1-k)b = b$.

Because f is continuous then by Intermediate Value Theorem there exists $x_0 \in [a, \infty)$ such that $f(x_0) = x_0$. The uniqueness of the fixed point of f can be easily seen since $x_0 - y_0 = f(x_0) - f(y_0) \leq k(x_0 - y_0)$ holds if and only if $x_0 = y_0$.

Case 2. $f(a) < a$

We claim that $f(x) < x$ for all $x \in [a, \infty)$. To show this we assume $f(x) \geq x$ for some $x > a$. Then $f(x) - f(a) \geq x - a > k(x - a)$ which is a contradiction.

Therefore the proof is completed.

Example 2. $f(x) = e^{-x}$, $x \geq 0$.

We see that f is differentiable and $f(0) = 1 \geq 0$ then by Mean Value Theorem we have $f(x) - f(y) = f'(c)(x - y) \leq \frac{1}{2}(x - y)$ for all x, $y \in \mathbb{R}$ and for some c lies in between x and y.

So f has a fixed point on $[0, \infty)$ as shown in figure 2.

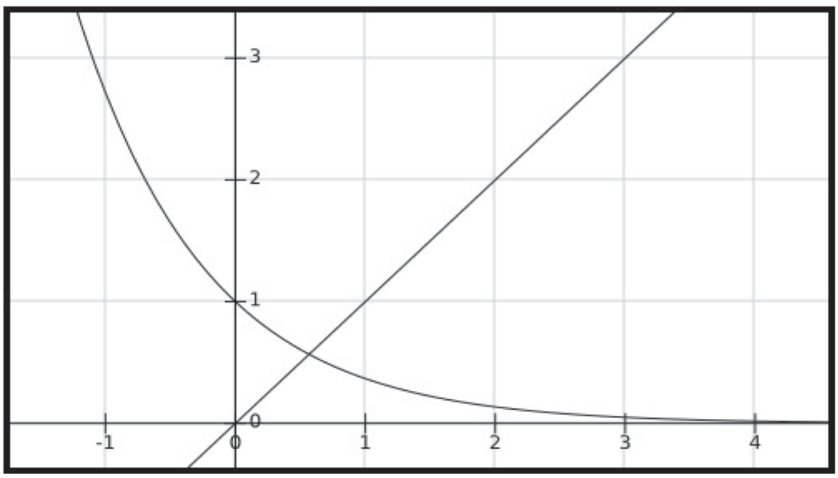

Figure 2

Example 3. Let $k<1$ and $g(x) = k$ In x, for $x \geq 1$.

Then g is differentiable and $g(1) = 0 < 1$. Again by Mean Value Theorem there exists $c > 1$ such that $g(x) - g(y) = g'(c)(x - y) = \frac{k}{c}(x - y) \leq \max\{0, k\}(x - y)$ for all x, $y > 1$.

Hence g has no fixed point by proposition 1 and the graph of f is shown in Figure 3 for the case $k = 0.9$.

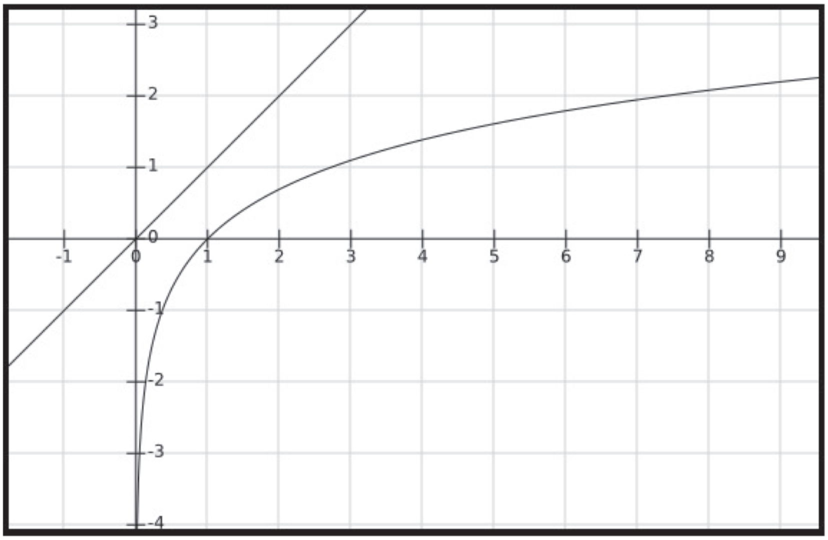

Figure 3

A direct consequence of the proposition 1 is the following corollary.

Corollary 1. *Let* $a \in \mathbb{R}$ and $k > 1$. *Suppose that* $f : [a, \infty) \to \mathbb{R}$ *is a continuous function satisfying*

$$f(x) - f(y) \geq k(x-y) \text{ for all } x, y \in [a, \infty).$$

Then either f *has a unique fixed point* or $f(x) > x$ *for all* $x \in [a, \infty)$.

Proof. Obviously, f is a strictly increasing function since $x > y$ implies.

$f(x) = f(x) - f(y) + f(y) \geq k(x-y) + f(y) > f(y)$

So f is one to one. Then we can define $g(x) = f^{-1}(x)$ for all $x \in [f(a), \infty)$.

Thus g is continuous and satisfies $g(x) - g(y) \leq \frac{1}{k}(x-y)$ for all $x, y \in [f(a), \infty)$.

By the Proposition 1, g either has a unique fixed point or $g(x) > x$ for all $x > f(a)$.

Hence f either has a unique fixed point or $f(x) < x$ for all $x \in [a, \infty)$. □

Example 4. Let $k > 1$ and $f(x) = e^{kx} - 2$, $x \geq 0$.

It is obvious that g does not satisfy the contraction but g has a fixed point on $[0, \infty)$ by checking the conditions in the corollary 1. That is $f(0) = -1 < 0$ and $f'(x) = ke^{kx} \geq k > 1$ for all $x > 0$.

Thus there exists $c > 0$ such that $f(x) - f(y) = f'(c)(x-y) \geq k(x-y)$ for all $x, y \geq 0$.

Hence f has a unique fixed point. The graph of f is shown in the figure 4 below for the case $k = 1.1$.

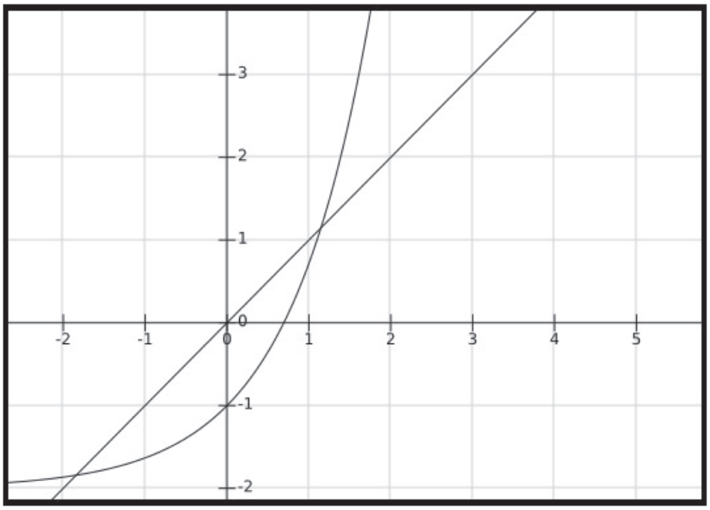

Figure 4

Example 5. $g(x) = x^4 - x + 2,\ x \geq 1.$

Since $g'(x) = 4x^3 - 1 \geq 3$ for $x \geq 1$ and $g(1) = 1 - 1 + 2 > 1$, the Mean Value Theorem implies that $g(x) - g(y) \leq 3\ (x - y)$ for all $x, y \geq 1$. Hence g has no fixed point for $x \geq 1$.

In fact, we can see that g has the minimum at $x = \dfrac{1}{\sqrt[3]{4}}$.

Thus $g(\dfrac{1}{\sqrt[3]{4}}) = \dfrac{1}{4\sqrt[3]{4}} - \dfrac{1}{\sqrt[3]{4}} + 2 > 1 > \dfrac{1}{\sqrt[3]{4}}$ and $g'(x) = 4x^3 - 1 > 0$ for $x > \dfrac{1}{\sqrt[3]{4}}$ and $g'(x) = 4x^3 - 1 < 0$ for $x > \dfrac{1}{\sqrt[3]{4}}$.

This implies that g actually has no fixed point in $\mathbb{R}$ as shown in figure 5.

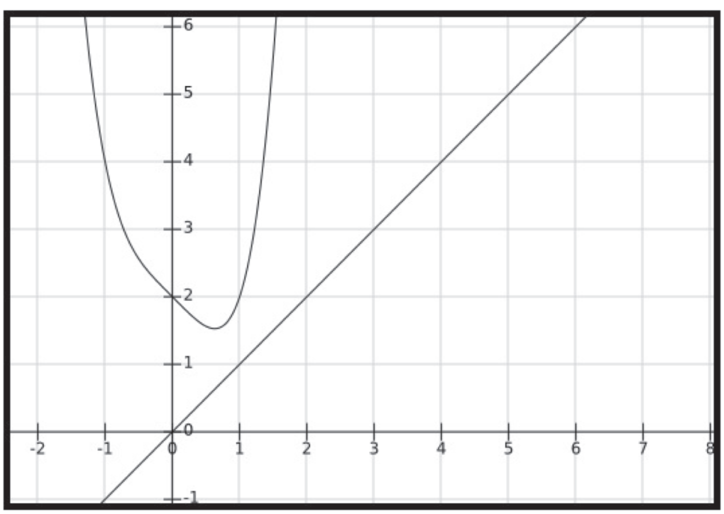

Figure 5

In proposition 1 and corollary 1 if we take $a = -\infty$ then we have the following corollaries.

Corollary 2. *Let $k < 1$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying*

$$f(x) - f(y) \leq k\ (x - y) \text{ for all } x,\ y \in \mathbb{R}.$$

Then f has a unique fixed point.

Proof. It suffices to show that there is $a \in \mathbb{R}$ such that $f(a) > a$. Again we prove this by a contradiction.

Suppose $f(x) \leq x$ for all $x \in \mathbb{R}$.

Then $f(x) = f(x) - f(y) + f(y) \leq k(x - y) + f(y) \leq kx + (1 - k)y$ for all $x,\ y \in \mathbb{R}$.

In particular, we have $f(0) \leq (1 - k)\ y$ for all $y \in \mathbb{R}$ implying that $f(0)$ is unbounded below which is a contradiction. Hence there is $a \in \mathbb{R}$ such that $f(a) > a$ and therefore f has a unique fixed point by the proposition 1. □

Corollary 3. Let $k > 1$. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying*

$$f(x) - f(y) \geq k(x - y) \text{ for all } x,\ y \in \mathbb{R}.$$

Then f has a unique fixed point.

Proof. Since f is one to one, we define a function $g(x) = f^{-1}(x)$, $x \in \mathbb{R}$.

Then g satisfies $g(x) - g(y) \leq \frac{1}{k}\ (x - y)$ for all $x,\ y \in \mathbb{R}$.

Hence by the corollary 2, g has a unique fixed point implying that f has also a unique fixed point. □

Conclusion

We have investigated the conditions that guarantee the existence of a fixed point of a continuous function. We have also provided simple proofs to verify a fixed point of continuous functions by analyzing the slope of functions. However, it does not provide where the fixed point is. A common way to check this is to find 2 real numbers a and b and check if the function has the opposite sign for those two real numbers. In the case of a contraction mapping, we have known from [2] that if we define $y_n = f^n(y)$, where f^n means f composites to itself n times, then y_n converges to a fixed point. The generalization of this result can also be found in [3]. Unfortunately, this property is lost when we weaken the contraction condition to the equation (1). For the corollary 2 and 3, it is quite surprising that the existence of a fixed point depends on the slope of functions that is lesser than a constant lesser than 1. This also implies the existence and uniqueness of a fixed point in the case of contraction mapping. For further exploration It would be an interesting topic to investigate this on more general metric spaces.

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