บทความวิจัย

Decoherence ของ Density Matrix ในลิมิตที่เวลามีค่ามาก ๆ

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บทคัดย่อ

เราได้แสดงให้เห็นว่า decoherence เกิดขึ้นได้โดยมีเพียงอันตรกิริยาในรูปแบบของ weak measurement แม้ว่าจะไม่มี environment interaction ภายหลังที่เราลดรูป density matrix สำหรับระบบผสมโดยการทำ partial trace แล้ว coherence จะถูกทำลายสำหรับลิมิตที่เวลามีค่ามาก ๆ ผลที่ได้นี้ เป็นสิ่งที่เหมือนกับ environment-induced decoherence เราวิเคราะห์การเสื่อมลงและนิยาม decoherence time scale ของ off-diagonal elements decoherence time scale นี้เป็นไปตามหลักความไม่แน่นอนของเวลาและพลังงาน สำหรับตัวอย่างเราได้แก้ปัญหา Stern-Gerlach experiment สำหรับการวัดสถานะสปิน

คำสำคัญ: decoherence, weak measurement, density matrix

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Decoherence of the Density Matrix in the Large Time Limit

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ABSTRACT

We present that decoherence is caused by only the weak measurement, although there is no environment interactions. After we reduce the density matrix for the composite system by using the partial trace, coherence is destroyed for large time limit. This result is as same as the environment-induced decoherence. We analyze decay and define the decoherence time scale for decaying of the off-diagonal elements. This decoherence time scale obeys the uncertainty relation for time and energy. For an example, we solve the Stern-Gerlach experiment for investigating the spin state.

Keywords: decoherence, weak measurement, density matrix

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Introduction

Quantum theory works successfully in practical application, some of its concepts seem debate when related to the world of our experience. For an example,the superposition principle plays the most central role in all considerations of standard quantum theory, and even the paradoxes of quantum mechanics. When the linear superposition of some basis states is obtained in a given measurement the wave function collapses to one of the basis state contained in the wave function. Though it can be express in the probable outcome, this process can not be discussed in mechanism for the collapse. In 1928 Bohr [1] postulated that the wave function collapse occurs when the quantum system comes into contact with an apparatus which must be described classically. In 1932 von Neumann [2] considered an irreversible reduction process taking the quantum superposition into a statistical mixture which is classically meaningful and interpretable. According to Bohm [3], this implies that the classical properties as we observe the more contained only as potentialities in the state vector. An interesting line of investigation to solve the problem of measurement is to treat on a Stern-Gerlach experiment, i.e. a measurement of spin-$\frac{1}{2}$ particle in the presence of an inhomogeneous magnetic field interacting with the environment. In 1985 Venugopalan, Kumar and Ghosh [4] proposed that decoherence is caused by the interaction in which the environment in effect monitors certain observable of the system, destroying coherence between the pointer states corresponding to their eigenvalues. Most studies of decoherence [5-9] in the literature deal with an environment modelled by a collection of oscillators, and the dynamics of the reduced density matrix of the system of interest is then studied via the corresponding master equation. Following the FV approach [10], the system and its environment have factorizable initial condition, i.e. decoupled at time $t = 0$. Hakim and Ambegaokar [11] generalized this choice, but their method is applicable only to systems for which the total Hamiltonian has translational invariance.

The paper is organized as follows. In the next section, two processes of evolution is given. In the third section, we show that a system-apparatus interaction in the weak measurement causes decoherence of the reduced density matrix at long time. In the fourth section, we consider the Stern-Gerlach experiment without the environment. The off-diagonal elements of the reduced density matrix decay to zero at large times. Finally in the last section, we summarize the results.

The Measuring Process

We consider the dynamics of quantum states in two processes.

(i) By following the postulates of quantum theory, states of quantum systems evolve according to the deterministic Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\left|\Psi\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \tag{1}$$

where H is a Hamiltonian of the closed system. The density matrix ρ_0 changes with time by the formula

$$\rho_0 \to \rho_t = U(t,t_0)\rho_0 U^{\dagger}(t,t_0), \tag{2}$$

where $U(t,t_0)$ and ρ_0 are respectively the time-evolution operator and initial density matrix of the quantum system. If the Hamiltonian operator is independent of time, then the time-evolution operator is given by

$$U(t,t_0) = \exp\left[-\frac{i}{\hbar}\mathbf{H}(t-t_0)\right]. \tag{3}$$

In this sense we can compute the state or the density matrix at an arbitrary time and the measurable properties of the system can be predicted probabilistically.

(ii) The collapse of the state in a quantum measurement is a process which destroys the linear superposition of the basis states. The defnitive outcome can only be accounted by following the von Neumann's postulate [2]. After the measurement, the wave function is one of the basis states contained in the wave function. By the notion of the collapse of the wave function, if the initial state of the quantum system is $\left|\Psi\right\rangle$ and the measurement gives the result a_m (an observable A is a Hermitain operator with $\mathbf{A}\left|a_m\right\rangle = a_m\left|a_m\right\rangle$) then the state of the quantum system is changed to be the state $\mathbf{P}_m\left|\Psi\right\rangle$ with the probability

$$p(m) = \left\langle\Psi\right|\mathbf{P}_m^{\dagger}\mathbf{P}_m\left|\Psi\right\rangle, \tag{4}$$

where $\mathbf{P}_m = \left|a_m\right\rangle\left\langle a_m\right|$ is the projector (the index m refers to the measurement outcome that may occur in the experiment). After the measurement, the density matrix ρ_0 is changed as

$$\rho_0 \to \rho = \sum_m \mathbf{P}_m^\dagger \rho_0 \mathbf{P}_m. \tag{5}$$

For an example, we consider the two-level system. By following the process (i), the density matrix evolves, according to eq.(2), to

$$\rho_t = U(t,t_0)\begin{bmatrix} |\sin\theta|^2 & \sin\theta\cos^*\theta \\ \sin^*\theta\cos\theta & |\cos\theta|^2 \end{bmatrix} U^\dagger(t,t_0), \tag{6}$$

where the initial state of the two-level system is $|S\rangle = \sin\theta|\uparrow\rangle + \cos\theta|\downarrow\rangle$ and $\rho_0 = |S\rangle\langle S|$. The Hamiltonian operator for the two-level system is $H = \frac{1}{2}\hbar\omega\sigma$ so that the time-evolution operator $U(t, t_0 = 0)$ can be written as

$$U(t,0) = e^{-\frac{i}{2}\omega\sigma t}, \tag{7}$$

where $\hbar\omega$ is the energy difference between the upper state $|\uparrow\rangle$ and lower state $|\downarrow\rangle$ and σ is the 2 x 2 (Pauli-spin) matrix notation

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{8}$$

Here $\sigma|\uparrow\rangle = |\uparrow\rangle$ and $\sigma|\downarrow\rangle = -|\downarrow\rangle$. By using eqs.(7) and (8), we can rewrite eq. (6) as

$$\rho_t = \begin{bmatrix} |\sin\theta|^2 & e^{-i\omega t}\sin\theta\cos^*\theta \\ e^{i\omega t}\sin^*\theta\cos\theta & |\cos\theta|^2 \end{bmatrix}. \tag{9}$$

Following the process (ii), the density matrix is changed by measurements in the form of eq.(5). Since the projection operators corresponding to the upper level $(\mathbf{P}_\uparrow)$ and the lower level $(\mathbf{P}_\downarrow)$ are

$$\mathbf{P}_\uparrow = |\uparrow\rangle\langle\uparrow| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{P}_\downarrow = |\downarrow\rangle\langle\downarrow| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{10}$$

the density matrix is

$$\rho = \begin{bmatrix} |\sin\theta|^2 & 0 \\ 0 & |\cos\theta|^2 \end{bmatrix}. \tag{11}$$

Notice that the arbitrary changes of the density matrix due to measurements given by eq.(5) leading to eq.(11) is different with the time-evolution of the density matrix given by eq.(9), i.e. $\rho \neq \rho_t$ or

$$\begin{bmatrix} |\sin\theta|^2 & 0 \\ 0 & |\cos\theta|^2 \end{bmatrix} \neq \begin{bmatrix} |\sin\theta|^2 & e^{-i\omega t}\sin\theta\cos^*\theta \\ e^{i\omega t}\sin^*\theta\cos\theta & |\cos\theta|^2 \end{bmatrix}. \tag{12}$$

Decoherence for Large Time Limit

We investigate the special process (i) which gives the result as same as the process (ii). By starting with the theory of measurement, we cannot observe the quantum system by itself, but must rather investigate the quantum system S interacting with the apparatus A. Let us consider the composite system, the quantum system and the apparatus. The total Hamiltonian of this composite system is written as

$$\mathbf{H}_{total} = \mathbf{H_S} + \mathbf{H_A} + \mathbf{H_I}, \tag{13}$$

where $\mathbf{H_S}$ represents the Hamiltonian of the quantum system, $\mathrm{H_A}$ that of the apparatus particle, $\mathbf{H_I}$ that of the coupling interaction which is assumed to be the weak measurement

$$\mathbf{H}_I = g(t)\mathbf{x}\mathbf{A}, \tag{14}$$

where $g(t) = g$ a coupling constant for $t > 0$ and 0 otherwise, $\mathbf{x}$ is the position operator of the apparatus $(\mathbf{x}|x\rangle = x|x\rangle)$ and A is some operators representing the measurement acting on the quantum system described by the eigenvalue equation

$$\mathbf{A}|a_m\rangle = a_m|a_m\rangle. \tag{15}$$

Here, we assume the non-degeneracy case.

The density matrix at time t depends on the initial density matrix at time $t = 0$. If we assume that the quantum system and the apparatus are decoupled at $t \leq 0$ (factorizable initial condition) and the initial wave function of the apparatus is a Gaussian wave packet while the initial state of the quantum system is an arbitrary state |quantum state⟩, then the initial state of the composite system projecting on the position state $|x_0\rangle$ of the apparatus is

$$\langle x_0|\Psi(t=0)\rangle = \left(\frac{1}{\sqrt{\alpha\pi}} e^{-\frac{x_0^2}{2\alpha^2} - \frac{i}{\hbar} p_0 x_0}\right) |\text{quantum state}\rangle \tag{16}$$

and the initial density matrix is

$$\rho_0 = \rho_{A0} \otimes \rho_{S0}, \tag{17}$$

where ρ_{S0} and ρ_{A0} are the initial density operator for the quantum and apparatus system respectively. For simplicity, we assume that the apparatus Hamiltonian $\mathbf{H_A}$ is a free particle

$$\mathbf{H_A} = \frac{\mathbf{P}^2}{2M}, \tag{18}$$

where M is a mass of the apparatus particle. The time-evolution operator is now

$$U(t, t_0 = 0) = \exp\left\{-\frac{i}{\hbar}(\mathbf{H}_S + \frac{\mathbf{P}^2}{2M} + g\mathbf{x}\mathbf{A})t\right\} \tag{19}$$

and the initial density matrix evolves in time by following the process (i) as

$$\rho_0 \rightarrow \rho_t = U(t, t_0)\rho_0 U^\dagger(t, t_0). \tag{20}$$

For the limitation on the measurement [12], Araki and Yanase [13] proved that an exact measurement of an operator which does not commute with a conserved quantity is impossible. So we consider by following this limitation that the observed operator satisfies a conservation law. It implies that the symmetry conservation principles hold, $[\mathbf{H}_S$, A]=0 (for the momentum operator $[\mathbf{H}_S$, P] = 0 i.e. homogeneity of space, for the total angular momentum $[\mathbf{H}_S$, L] = 0 i.e. isotropy of space). By using the Baker-Hausdorf formula $e^{(\mathbf{A}+\mathbf{B})} = e^{\mathbf{A}} e^{\mathbf{B}} e^{-\frac{1}{2}[\mathbf{A},\mathbf{B}]}$ for $[[\mathbf{A},\mathbf{B}],\mathbf{A}] = [[\mathbf{A},\mathbf{B}],\mathbf{B}] = 0$, eq. (20) can be written as

$$\rho_t = e^{-\frac{i}{\hbar}\mathbf{H}_S t}\left[e^{-\frac{i}{\hbar}(\frac{\mathbf{P}^2}{2M}+g\mathbf{x}\mathbf{A})t}\rho_0 e^{\frac{i}{\hbar}(\frac{\mathbf{P}^2}{2M}+g\mathbf{x}\mathbf{A})t}\right]e^{\frac{i}{\hbar}\mathbf{H}_S t}. \tag{21}$$

We insert the completeness relation $I = \int dx|x\rangle\langle x|$ on the left and right hand of ρ_0. We obtain

$$\rho_t = e^{-\frac{i}{\hbar}\mathbf{H}_S t}\int dx''\int dx'\int dx_0\int dx_0'|x''\rangle K(x'',x_0';t|\mathbf{A})\rho_{S0}\langle x_0'|\rho_{A0}|x_0\rangle K^\dagger(x',x_0;t|\mathbf{A})\langle x'|e^{\frac{i}{\hbar}\mathbf{H}_S t}, \tag{22}$$

where $K(x'',x_0';t|\mathbf{A}) = \langle x''|e^{-\frac{i}{\hbar}(\frac{\mathbf{P}^2}{2M}+g\mathbf{A}\mathbf{x})t}|x_0'\rangle$. We now see that there is the propagator K for the motion of a particle in a linear potential $g\mathbf{A}\mathbf{x}$,, depending on the observed operator A. This propagator has been recently discussed by several authors [14-17] using different techniques. The formula is

$$\begin{aligned} K(x'',x_0';t|\mathbf{A}) &= \langle x''|e^{-\frac{i}{\hbar}(\frac{\mathbf{P}^2}{2M}+g\mathbf{A}\mathbf{x})t}|x_0'\rangle \\ &= \left(\frac{M}{2\pi i\hbar t}\right)^{\frac{1}{2}}\exp\left\{\frac{i}{\hbar}\left[\frac{M}{2t}(x''-x_0')^2+\frac{g\mathbf{A}t}{2}(x''+x_0')-\frac{g^2\mathbf{A}^2t^3}{24M}\right]\right\} \end{aligned} \tag{23}$$

and

$$\begin{aligned} K(x',x_0;t|\mathbf{A}) &= \langle x'|e^{-\frac{i}{\hbar}(\frac{\mathbf{P}^2}{2M}+g\mathbf{A}x)t}|x_0\rangle \\ &= \left(\frac{M}{2\pi i\hbar t}\right)^{\frac{1}{2}}\exp\left\{\frac{i}{\hbar}\left[\frac{M}{2t}(x'-x_0)^2+\frac{g\mathbf{A}t}{2}(x'+x_0)-\frac{g^2\mathbf{A}^2t^3}{24M}\right]\right\}. \end{aligned} \tag{24}$$

For the analysis of subsystem of the composite quantum system, we consider the partial trace over the apparatus system. The partial trace is defined by

$$\rho_{st} = \mathrm{Tr}_{ap}[\rho_t] \equiv \int dx\langle x|\rho_t|x\rangle. \tag{25}$$

By inserting eq.(16), eq.(23) and eq.(24) into eq.(22) and using eq.(25), it is obvious that the reduced density operator for the quantum system is

$$\rho_{st} = e^{-\frac{i}{\hbar}\mathbf{H}_S t}\int dx\Psi(x,t|\mathbf{A})\rho_{S0}\Psi^\dagger(x,t|\mathbf{A})e^{\frac{i}{\hbar}\mathbf{H}_S t}, \tag{26}$$

where, for convenient, we define the wave function for the apparatus system at time t which contains the interaction between the quantum system and the apparatus in the form of the linear potential propagator

$$\Psi(x,t|\mathbf{A}) \equiv \int dx_0' K(x,x_0';t|\mathbf{A}) \left[\frac{1}{\sqrt{\alpha\pi}} e^{-\frac{x_0'^2}{2\alpha^2} - \frac{i}{\hbar} p_0 x_0'}\right]. \quad (27)$$

The integral eq.(27) can be evaluated by completing the square in the exponent. One simply obtains

$$\Psi(x,t|\mathbf{A}) = \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{1}{\alpha^2 + \frac{i\hbar t}{M}}} \exp\left[-\frac{i}{\hbar}\left(\frac{M\alpha^2}{2t} \frac{\left(\frac{p_0 t}{M} - x + \frac{1}{2M} g\mathbf{A}t^2\right)^2}{\alpha^2 + \frac{i\hbar t}{M}} + \left(\frac{Mx^2}{2t} + g\mathbf{A}xt + \frac{g^2\mathbf{A}^2 t^3}{24M}\right)\right)\right]. \quad (28)$$

So the position distribution is

$$|\Psi(x,t|\mathbf{A})|^2 = \left[\frac{1}{\pi}\sqrt{\frac{\alpha^2}{\alpha^4 + \frac{\hbar^2 t^2}{M^2}}}\right] \exp\left\{\left(-\frac{1}{\alpha^4 + \frac{\hbar^2 t^2}{\alpha^2 M^2}}\right)\left[x - \left(\frac{p_0 t}{M} + \frac{1}{2M} g\mathbf{A}t^2\right)\right]^2\right\}. \quad (29)$$

For the different eigenvalue, the pointer wave packet would spread in time in the position space. We have the separable distance between their peaks which point the eigenvalues a_m and a_{m+1} as,

$$\mathbf{\Delta}_x = \frac{1}{2M} g \mathbf{\Delta}_{\mathbf{A}} t^2; \;\; \mathbf{\Delta}_{\mathbf{A}} = |a_m - a_{m+1}|. \quad (30)$$

This is a separate distance between the pointer corresponding with the observed results a_m and a_{m+1}. The wave function and the position distribution are functions of the observed operator A, acting on the initial density matrix of the quantum system. By inserting the identity $I = \sum_m |a_m\rangle\langle a_m|$ in the left and right of ρ_{S0} in eq.(26) and using the Gaussian integrals,

$$\begin{aligned} &\int dx \Psi(x,t|a_n)\Psi^\dagger(x,t|a_m) \\ = \;& \left[\frac{1}{\pi}\sqrt{\frac{\alpha^2}{\alpha^4 + \frac{\hbar^2 t^2}{M^2}}}\right] \int dx \exp\left[-\frac{i}{\hbar}(gxt)(a_n - a_m) - \frac{i}{\hbar}\left(\frac{g^2 t^3}{24M}\right)\left(a_n^2 - a_m^2\right)\right] \\ & \times \exp\left[-\frac{i}{\hbar}\left(\frac{M\alpha^2}{2t}\right)\left(\frac{\left(\frac{p_0 t}{M} - x + \frac{1}{2M} g a_n t^2\right)^2}{\alpha^2 + \frac{i\hbar t}{M}} - \frac{\left(\frac{p_0 t}{M} - x + \frac{1}{2M} g a_m t^2\right)^2}{\alpha^2 - \frac{i\hbar t}{M}}\right)\right] \\ = \;& \exp\left[-\left(\frac{g\alpha t}{4\hbar}\right)^2 (a_n - a_m)^2\right] \exp\left[-\frac{i}{\hbar}\left(\frac{g^2 t^3}{24M}\right)\left(a_n^2 - a_m^2\right)\right], \end{aligned} \quad (31)$$

the reduced density matrix for the quantum system is

$$\rho_{st} = e^{-\frac{i}{\hbar}\mathbf{H}_S t} \sum_n \sum_m |a_n\rangle \langle a_m| \langle a_n|\rho_{s0}|a_m\rangle \left(e^{-\frac{i}{\hbar}\left(\frac{g^2 t^3}{24M}\right)(a_n^2 - a_m^2)} G_{nm}(t) \right) e^{\frac{i}{\hbar}\mathbf{H}_S t}, \tag{32}$$

where $G_{nm}(t)$ is defined as

$$G_{nm}(t) = \exp\left[-\left(\frac{\alpha g t}{4\hbar}\right)^2 (a_n - a_m)^2 \right]. \tag{33}$$

For large time limit, the function $G_{nm}(t)$ decay to zero for the off-diagonal elements thus the decoherence corresponding to the decay of the element correlated with the off-diagonals in the reduced density matrix (Since $[\mathbf{H}_S, \mathbf{A}] = 0$, $G_{nm} = 0\ \forall m \neq n$ implies that $\langle a_m|\rho_{st}|a_n\rangle = 0\ \forall m \neq n$ i.e. ρ_{st} is the diagonal matrix). We define the decoherence time scale which the reduced density operator have decayed to $1/e$ of their original values for the off-diagonal elements as

$$\tau_d = \frac{4\hbar}{\alpha g \Delta a_{nm}}, \tag{34}$$

where $\Delta a_{nm} = |a_n - a_m|$ is the differential eigenvalue of operator A. This definition obeys the uncertainty relation for time and energy

$$\Delta E \Delta t \geq \hbar, \tag{35}$$

where the energy uncertainty of the total energy ΔE is equal to the uncertainty of the perturbation energy $\mathbf{H_I} = gx\mathbf{A}$. Here, ΔA corresponds to Δa_{mn} and the uncertainty in time Δt corresponds to the decoherence time scale τ_d.

After $t \gg \tau_d$, the density matrix ρ_{st} becomes

$$\rho_{st} = e^{-\frac{i}{\hbar}\mathbf{H}_S t} \left[\sum_n \langle a_n|\rho_{S0}|a_n\rangle P_n \right] e^{\frac{i}{\hbar}\mathbf{H}_S t} \tag{36}$$

where $\mathbf{P}_n = |a_n\rangle \langle a_n|$ is the projection operator comesponding to the state $|a_n\rangle$.

Now we obtain decoherence of the density matrix by discussing the time evolution of the initial density matrix of the composite system. By eliminating the degree of freedom of the apparatus using the partial trace, the weak measurement interaction leads to decoherence in the reduced density matrix. We can see in eq.(33) that the reduced density matrix ρ_{st} becomes the diagonal one for the large time limit. This result is the same as environment-induced decoherence (see, for example, in [9]). Without an environment, coherence can also be destroyed by the coupling interaction in the form of weak measurement between the observed system and the apparatus.

The Stern-Gerlach Experiment

In this section, we consider the Stern-Gerlach experiment for the measurement on a quantum system (spin-$\frac{1}{2}$ system) with a macroscopic quantum apparatus (the particle trajectory). We start with the pure density matrix of the system-apparatus which have the coupling interaction in the form of weak measurement. When the pure density matrix is reduced and the large time limit is considered the of-diagonal elements decay to zero.

A general Hamiltonian for the Stern-Gerlach experiment, that applies to spin $\frac{1}{2}$ but can be straightforwardly extended to any spin, is

$$\mathbf{H}_{total} = \mathbf{H_A}+\mathbf{H_S}+\mathbf{H_{AS}} = \frac{\mathbf{P}^2}{2m} + \lambda\sigma_z + \varepsilon\mathbf{x}\sigma_z, \quad (37)$$

where $\mathbf{x}$ and $\mathbf{P}$ are position and momentum operators of the apparatus particle. The operator σ_z is to measure the z component of spin which has the eigenvalues $1=:\sigma_1$ and $-1=:\sigma_2$ with the corresponding eigenstates $|\sigma_z,1\rangle$ and $|\sigma_z,2\rangle$ respectively (in the previous notation, $|\sigma_z,1\rangle = |\uparrow\rangle$ and $|\sigma_z,2\rangle = |\downarrow\rangle$).

If the initial condition for this system is a product of a Gaussian wave packet and a general spin state i.e.

$$\langle x|\Psi_0\rangle = \left(\frac{1}{\sqrt{\alpha\sqrt{\pi}}}e^{\frac{i}{\hbar}p_0x-\frac{x^2}{2\alpha^2}}\right)|S\rangle, \quad (38)$$

then the initial density matrix can be written as

$$\langle x|\rho_0|x'\rangle = \left(\frac{1}{\sqrt{\alpha\sqrt{\pi}}}e^{\frac{i}{\hbar}p_0x-\frac{x^2}{2\alpha^2}}\right)|S\rangle\langle S|\left(\frac{1}{\sqrt{\alpha\sqrt{\pi}}}e^{-\frac{i}{\hbar}p_0x'-\frac{x'^2}{2\alpha^2}}\right). \quad (39)$$

Here, a spin state $|S\rangle = \sin\theta|\sigma_z,1\rangle + \cos\theta|\sigma_z,2\rangle$. The time evolution of this density matrix according to process (i) in the second section is

$$\rho_t = U(t,t_0)\rho_0U^\dagger(t,t_0), \quad (40)$$

where the unitary evolution operator is

$$U(t,t_0=0) = e^{-\frac{i}{\hbar}(\frac{\mathbf{P}^2}{2m}+\lambda\sigma_z+\varepsilon\mathbf{x}\sigma_z)t}. \quad (41)$$

By using the Baker-Hausdorf formula $e^{(\mathbf{A}+\mathbf{B})} = e^{\mathbf{A}}e^{\mathbf{B}}e^{-\frac{1}{2}[\mathbf{A},\mathbf{B}]}$ for $[[\mathbf{A},\mathbf{B}],\mathbf{A}] = [[\mathbf{A},\mathbf{B}],\mathbf{B}] = \mathbf{0}$, the unitary evolution operator can be rewritten as

$$U(t,t_0) = e^{-\frac{i}{\hbar}(\lambda\sigma_z)t}e^{-\frac{i}{\hbar}(\frac{\mathbf{P}^2}{2m}+\varepsilon\mathbf{x}\sigma_z)t} \quad (42)$$

and

$$U^\dagger(t,t_0) = e^{\frac{i}{\hbar}(\frac{\mathbf{P}^2}{2m}+\varepsilon\mathbf{x}\sigma_z)t}e^{\frac{i}{\hbar}(\lambda\sigma_z)t} \quad (43)$$

for $\left[\frac{\mathbf{P}^2}{2m},\lambda\sigma_z\right] = 0$ and $[\lambda\sigma_z,\varepsilon\mathbf{x}\sigma_z] = 0$.

We now, as in the previous section, consider the effective system by using the partial trace over the apparatus system. Form eqs.(32) and (33), the reduced density operator for the spin system in this Stern-Gerlach experiment is

$$\rho_{st} = \sum_{i=1}^{2}\sum_{j=1}^{2} e^{-\frac{i}{\hbar}(\lambda\sigma_i)t}|\sigma_z,i\rangle\langle\sigma_z,j|\langle\sigma_z,i|\rho_{S0}|\sigma_z,j\rangle\left[e^{-\frac{i}{\hbar}\left(\frac{\varepsilon^2t^3}{24M}\right)(\sigma_i^2-\sigma_j^2)}G_{ij}(t)\right]e^{\frac{i}{\hbar}(\lambda\sigma_j)t}, \quad (44)$$

where $\rho_{S0} = |S\rangle\langle S|$ and

$$G_{ij}(t) = \exp\left[-\left(\frac{\alpha\varepsilon t}{4\hbar}\right)^2(\sigma_i-\sigma_j)^2\right]. \quad (45)$$

In the large time limit, the function $G_{ij}(t)$ tends to

$$G_{ij}(t) = \begin{cases} 1 & ; \quad i=j \\ 0 & ; \quad \text{otherwise} \end{cases} \quad (46)$$

and the off-diagonal elements decay to zero thus the reduced density matrix becomes

$$\rho_{st} = \sum_{i=1}^{2} |\sigma_z, i\rangle \langle\sigma_z, i| \langle\sigma_z, i| \rho_{S0} |\sigma_z, i\rangle \tag{47}$$

or

$$\rho_{st} = \sum_{i=1}^{2} \mathbf{P}_i \langle\sigma_z, i| \rho_{S0} |\sigma_z, i\rangle \tag{48}$$

where $\mathbf{P}_i = |\sigma_z, i\rangle \langle\sigma_z, i|$ is the projection operator corresponding to the state $|\sigma_z, i\rangle, i = 1, 2$. In the matrix representation, eq.(48) can be written as

$$\rho_{st} = \begin{bmatrix} |\sin\theta|^2 & 0 \\ 0 & |\cos\theta|^2 \end{bmatrix}. \tag{49}$$

Note that this result is the same as eq.(11) caused by measurements (eq.(5) of process (ii)).

Conclusion

We have show that decoherence can appear when without existing environment by using the coupling interaction between a quantum system and an apparatus in form of the weak measurement. We start with the composite system containing the observed system and the apparatus system. Following the limitation on the measurement and the Baker-Hausdorf formula, the time-evolution operator can be separable in the form of Eq.(21). Writing the time-evolution operator as the form of eq.(21) yields a propagator of the apparatus system modifies by the linear potential $gx\mathbf{A}$. After we reduce the density matrix of the composite system by eliminating the degree of freedom of the apparatus system, coherence is destroyed in the large time limit. The off-diagonal elements decay at the rate $\tau_d^{-1} \sim 4\hbar/\alpha g \Delta a_{nm}$. The decoherence time scale in the form eq.(34) is a duration which the off-diagonal of the density matrix have decayed to $1/e$ of their original values. It satisfies the uncertainty relation f or time and energy. For example, we consider the Stern-Gerlach experiment for investigating the measurement of spin in the non-effect of environment. Following our modal the off-diagonal elements of the spin density matrix decay to zero for large time limit.

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References

1. Bohr, N. 1928. The Quantum Postulate and Recent Development of Atomic Theory. Nature 121: 580.
2. Von Neumann, J. 1932. Mathematische Grundlagen der Quantenmechanik (Springer-Verlag, Berlin); English translation by Beyer, R. T. 1955. Princeton. Princeton University Press. p. 417.
3. Bohm, D. 1951. Quantum Theory. Prentic-Hall. INC. p. 583.
4. Feynman, R. P. and Vernon, F. L. 1963. The Theory of a General Quantum System Interacting with a Linear Dissipative System. *Annales de Physique* 42: 118-173.
5. Hakim, V. and Ambegaokar, V. 1985. Quantum Theory of a Free Particle Interacting with a Linearly Dissipative Environment. *Physical Review A* 32: 423.
6. Venugopalan, A., Kumar, D. and Ghosh, R. 1995. Environment-Induced Decoherence I. The Stern-Gerlach Measurement. Physical A 220: 563.
7. Venugopalan, A. 1997. Decoherence and Schrödinger-Cat States in a Stern-Gerlach-Type Experiment. *Physical Review A* 56: 4307.
8. Banerjee, S. and Ghosh, R. 2000. Quantum Theory of a Stern-Gerlach System in Contact with a Linearly Dissipative Environment. *Physical Review A* 62: 042105.
9. Dekker, H. 1981. Classical and Quantum-Mechanics of the Damped Harmonic-Oscillator. *Physical Report* 80: 1.
10. Agarwal, G. S. 1971. Brownian Motion of a Quantum Oscillator. *Physical Review A* 4: 739.
11. Caldeira, A. O. and Leggett, A. J. 1983. Path Integral Aapproach to Quantum Brownian Motion. *Physica A*121A: 587.
12. Wigner, E. P. 1952. Die Messumg Quantenmechanischer Operatorem. *Zeitschrift für Physik* 131: 101.
13. Araki, H. and Yanase, M. M. 1960. Measurement of Quantum Mechanical Operators. *Physical Review* 120: 622.
14. Holstein, B. R. 1997. Physical and Unphysical Phases of Uniformly Accelerated Particles. American *Journal Physical* 65: 414.
15. Arrighini, G. P., Durante, N. L. and Guidotti, C. 1996. More on the Quantum Propagator of a Particle in a Linear Potential. American *Journal Physical* 64: 1036.

16. Robinett, R. W. 1996. Visualizing the Solutions of the Circular Infinite Well in Classical and Quantum Mechanics Visualizing the Solutions of the Circular Infinite Well in Classical and Quantum Mechanics.American *Journal Physical* 64: 803.
17. Brown, L. S. and Zhang, Y. 1994. Path Integral for the Motion of a Particle in a Linear Potential. *American Journal Physical* 62: 806.

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