บทความวิจัย

คำตอบและเสถียรภาพของบางสมการ

ศุภโชค อิสริยปาลกุล* และ วัชรพล พิมพ์เสริฐ

บทคัดย่อ

เราทำการหาคำตอบและเสถียรภาพของสมการเชิงฟังก์ชันสองสมการที่คล้ายกับสมการเชิงฟังก์ชันเดวิสัน ซึ่งก็คือ

$$f(xy)+mf(x+y)=f(xy+x)+f(my)$$

และ

$$f(xy)+mf(x+y)=f(xy+x)+mf(y)$$

เมื่อ $x, y \in \mathbb{R}$ และ $m \in \mathbb{R} \setminus \{0,1\}$ เป็นค่าคงที่ใดๆ

คำสำคัญ: สมการเชิงฟังก์ชัน สมการเชิงฟังก์ชันเดวิสัน เสถียรภาพ

ภาควิชาคณิตศาสตร์ มหาวิทยาลัยเกษตรศาสตร์

*ผู้นิพนธ์ประสานงาน, e-mail: kalamung05@hotmail.com

Solutions and Stabilities of Some Equations

Supachoke Isariyapalakul* **and Watcharapon Pimsert**

ABSTRACT

We find the solutions and stabilities of two functional equations which are a generalized version of Davison functional equation, i.e.,

$$f(xy) + mf(x+y) = f(xy+x) + f(my)$$

and

$$f(xy) + mf(x+y) = f(xy+x) + mf(y)$$

where $x, y \in \mathbb{R}$ and $m \in \mathbb{R} \setminus \{0,1\}$ is a constant.

Keywords: functional equation, Davison functional equation, stability

Department of Mathematics, Faculty of Science, Kasetsart University

*Corresponding author, e-mail: kalamung05@hotmail.com

1. Introduction

In 1940, Ulam [1] introduced the following problem, which has since been referred to as a "stability" problem: let f be a mapping from a group $(G_1,+)$ to a metric group $(G_2,+)$ with metric $d(.,.)$ such that

$$d(f(x+y), f(x)+f(y)) \leq \varepsilon.$$

Do there exist a group homomorphism $L: G_1 \to G_2$ and a constant $\delta_\varepsilon > 0$ such that $d(f(x), L(x)) \leq \delta_\varepsilon$ for all $x \in G_1$? This means that if we change a bit of the functional equation, then there is a little effect to its solution? In 1941, Hyers [2] proved that if $f: E_1 \to E_2$ is a function satisfying

$$\| f(x+y) - f(x) - f(y) \| \leq \delta$$

for all $x, y \in E_1$, where E_1 and E_2 are Banach spaces and δ is a given positive number, then there exists a unique additive function $T: E_1 \to E_2$ such that

$$\| f(x) - T(x) \| \leq \delta$$

for all $x \in E_1$. If f is a real continuous function on $\mathbb{R}$ satisfying

$$| f(x+y) - f(x) - f(y) | \leq \delta,$$

it was shown by Hyers and Ulam that there exists a constant k such that

$$| f(x) - kx | \leq 2\delta.$$

In 1980, T.M.K. Davison [3] introduced the functional equation

$$f(xy) + f(x+y) = f(xy+x) + f(y) \qquad (*)$$

in the 17^{th} ISFE (Oberwolfach). During the meeting, W. Benz presented that every continuous solution $f: \mathbb{R} \to \mathbb{R}$ of (*) for all $x, y \in \mathbb{R}$ is of the form $f(x) = ax + b$ where a,b are real constants. Next, in 1999, Jung and Sahoo [2] found the stability of (*) and its Pexider form:

$$f(xy) + g(x+y) = h(xy+x) + k(y). \qquad (**)$$

In 2000, $\mathbb{R}$. Girgensohn and K. Lajkó [4] solved the general solution of (*) and (**) for $x, y \in \mathbb{R}$ and for $x, y \in \mathbb{R}^+$, respectively.

In this paper, we propose the general solutions and stabilities of two functional equations which are an extended version of (*). Those are the functional equations

$$f(xy) + mf(x + y) = f(xy + mx) + f(my) \tag{1}$$

and

$$f(xy) + mf(x + y) = f(xy + mx) + mf(y) \tag{2}$$

where $x, y \in \mathbb{R}$ and $m \in \mathbb{R} \setminus \{0,1\}$.

2. Solutions

We find the solutions of (1) and (2), the results are

Theorem 2.1 *For a fixed $m \in \mathbb{R} \setminus \{0,1\}$. The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (1) for all $x, y \in \mathbb{R}$ if and only if f is an additive function such that f(mx) = mf(x).*

Proof. For a fixed $m \in \mathbb{R} \setminus \{0,1\}$. Suppose that f is an additive function such that $f(mx) = mf(x)$. Thus, we obtain

$$\begin{aligned} f(x,y) + mf(x + y) &= f(xy) + f(mx + my) \\ &= f(xy) + f(mx) + f(my) \\ &= f(xy + mx) + f(my), \end{aligned}$$

for all $x, y \in \mathbb{R}$. So f satisfies (1).

Next, we show the "if part" of this theorem. Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (1) for all $x, y \in \mathbb{R}$.

Replacing y by $y + m$ into (1), we get

$$f(xy + mx) + mf(x + y + m) = f(xy + 2mx) + f(my + m^2). \tag{2.1}$$

By adding (1) and (2.1), we have

$$f(xy) + mf(x + y) + mf(x + y + m) = f(my) + f(xy + 2mx) + f(my + m^2).$$

From the above equation, we substitute x by $\frac{x}{2}$ and y by $2y$:

$$f(xy) + mf(\tfrac{x}{2} + 2y) + mf(\tfrac{x}{2} + 2y + m) = f(2my) + f(xy + mx) + f(2my + m^2). \tag{2.2}$$

Next, we subtract (1) from (2.2), that is, (2.2) – (1), to get

$$mf(\frac{x}{2}+2y)+mf(\frac{x}{2}+2y+m)-mf(x+y)=f(2my)+f(2my+m^2)-f(my).$$

Then, replacing x by $x-y$ in the above equation, we get

$$mf(\frac{x}{2}+\frac{3y}{2})+mf(\frac{x}{2}+\frac{3y}{2}+m)-mf(x)=f(2my)+f(2my+m^2)-f(my).$$

Replacing y by $\frac{y}{3}$ in the last equation:

$$mf(\frac{x}{2}+\frac{y}{2})+mf(\frac{x}{2}+\frac{y}{2}+m)-mf(x)=f(\frac{2my}{3})+f(\frac{2my}{3}+m^2)-f(\frac{my}{3}).$$

Now, it is of the pexider form:

$$A_3(x+y)=A_2(x)+A_1(y)$$

where

$$A_1(t):=f(\frac{2mt}{3})+f(\frac{2mt}{3}+m^2)-f(\frac{mt}{3}), \tag{2.3}$$

$$A_2(t):=mf(t), \tag{2.4}$$

$$A_3:=mf(\frac{t}{2})+mf(\frac{t}{2}+m). \tag{2.5}$$

So there exists an additive function $A:\mathbb{R}\to\mathbb{R}$ and constants $d_1, d_2\in\mathbb{R}$ such that

$$A_1(x)=A(x)+d_1, \tag{2.6}$$

$$A_2(x)=A(x)+d_2, \text{ and} \tag{2.7}$$

$$A_3(x)=A(x)+d_1+d_2. \tag{2.8}$$

From (2.4) and (2.7), we get

$$f(x)=\frac{1}{m}A_2(x)=\frac{1}{m}A(x)+\frac{d_2}{m}. \tag{2.9}$$

From (2.3) and (2.6), we have

$$A(x)+d_1=f(\frac{2mx}{3})+f(\frac{2mx}{3}+m^2)-f(\frac{mx}{3}). \tag{2.10}$$

Hence, from (2.9) and (2.10), we obtain that

$$A(x)+d_1 = \frac{1}{m}A(\frac{2mx}{3})+\frac{d_2}{m}+\frac{1}{m}A(\frac{2mx}{3}+m^2)+\frac{d_2}{m}-\frac{1}{m}A(\frac{mx}{3})-\frac{d_2}{m}$$

$$= \frac{1}{m}A(mx)+\frac{1}{m}A(m^2)+\frac{d_2}{m}. \quad (2.11)$$

From (2.5), (2.8) and (2.9), we get

$$A(x)+d_1+d_2 = mf(\frac{x}{2})+mf(\frac{x}{2}+m)$$
$$A(x)+d_1+d_2 = A(\frac{x}{2})+d_2+A(\frac{x}{2}+m)+d_2$$
$$A(x)+d_1 = A(x)+A(m)+d_2$$
$$d_1 = A(m)+d_2. \quad (2.12)$$

Next, we substitute x by $-m$ in (2.11):

$$A(-m)+d_1 = \frac{1}{m}A(-m^2)+\frac{1}{m}A(m^2)+\frac{d_2}{m}$$

$$mA(-m)+md_1 = d_2$$
$$m(-A(m)+d_1) = d_2.$$

By (2.12), we obtain

$$md_2 = d_2$$
$$(m-1)d_2 = 0.$$

Since $m \neq 1$, so

$$d_2 = 0. \quad (2.13)$$

And replacing x by 0 in (2.11), we have

$$A(0)+d_1 = \frac{1}{m}A(0)+\frac{1}{m}A(m^2)+\frac{d_2}{m}$$

$$md_1 = A(m^2)+d_2.$$

From (2.12) and (2.13), so

$$mA(m) = A(m^2). \tag{2.14}$$

By using (2.11), (2.12), (2.13) and (2.14), we obtain that

$$\begin{aligned} A(x)+d_1 &= \frac{1}{m}A(mx)+\frac{1}{m}A(m^2)+\frac{d_2}{m} \\ mA(x)+md_1 &= A(mx)+A(m^2) \\ mA(x)+mA(m) &= A(mx)+mA(m) \\ mA(x) &= A(mx). \end{aligned}$$

So, by (2.9), we get

$$f(x)=\frac{1}{m}A(x)$$

where $A:\mathbb{R}\to\mathbb{R}$ is an additive function such that
$f(mx) = \frac{1}{m}A(mx) = \frac{1}{m}\, mA(x) = A(x) = mf(x)$, i.e., f is an additive function where $f(mx) = mf(x)$.

Theorem 2.2 *For a fixed $m\in\mathbb{R}\setminus\{0,1\}$. If the function $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation (2) for all $x, y\in\mathbb{R}$, then f is of the form*

$$f(x) = A(x) + b$$

where $A:\mathbb{R}\to\mathbb{R}$ is an additive function and $b\in\mathbb{R}$ is a constant.

Proof. Suppose that $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation (2) for all $x, y\in\mathbb{R}$. First, we substitute y by $y + m$ in (2):

$$f(xy + mx) + mf(x + y + m) = f(xy + 2mx) + mf(y + m). \tag{2.15}$$

By adding (2) and (2.15), we have

$$f(xy) + mf(x + y) + mf(x + y + m) = mf(y) + f(xy + 2mx) + mf(y + m).$$

Replacing x by $\frac{x}{2}$ and y by $2y$ in the last equation, we get

$$f(xy) + mf(\tfrac{x}{2} + 2y) + mf(\tfrac{x}{2} + 2y + m) = mf(2y) + f(xy + mx) + mf(2y + m). \tag{2.16}$$

From (2) and (2.16), we get

$$mf(\frac{x}{2}+2y)+mf(\frac{x}{2}+2y+m)-mf(x+y)=mf(2y)+mf(2y+m)-mf(y)$$
$$f(\frac{x}{2}+2y)+f(\frac{x}{2}+2y+m)-f(x+y)=f(2y)+f(2y+m)-f(y).$$

Next, replacing x by $x-y$:

$$f(\frac{x}{2}+\frac{3y}{2})+f(\frac{x}{2}+\frac{3y}{2}+m)-f(x)=f(2y)+f(2y+m)-f(y).$$

By substituting y by $\frac{y}{3}$ in the above equation, we obtain

$$f(\frac{x}{2}+\frac{y}{2})+f(\frac{x}{2}+\frac{y}{2}+m)=f(x)+f(\frac{2y}{3})+f(\frac{2y}{3}+m)-f(\frac{y}{3}).$$

We see that the above equation is of the pexider form

$$A_3(x+y)=A_2(x)+A_1(y)$$

where

$$A_1(t):=f(\frac{2t}{3})+f(\frac{2t}{3}+m)-f(\frac{t}{3}),$$
$$A_2(t):=f(t),$$
$$A_3(t):=f(\frac{t}{2})+f(\frac{t}{2}+m).$$

So the general solutions are

$$A_1(t):=A(t)+a,$$
$$A_2(t):=A(t)+b \qquad \text{and}$$
$$A_3(t):=A(t)+a+b$$

where $A:\mathbb{R}\to\mathbb{R}$ is an additive function and a,b are constants. So

$$f(x)=A(x)+b$$

3. Stabilities

In this section, we consider the stabilities of (1) and (2) with new condition, i.e., m is a nonzero real number. The results are stated below.

Theorem 3.1 *For a fixed m, $\delta\in\mathbb{R}$ with $m\neq 0$ and $\delta>0$. If the function $f:\mathbb{R}\to\mathbb{R}$ satisfies the inequality*

$$|f(xy) + mf(x + y) - f(xy + mx) - f(my)| \le \delta \tag{3.1}$$

for all $x, y \in \mathbb{R}$, *then there exists a unique additive function* $A: \mathbb{R} \to \mathbb{R}$ *such that*

$$\left| f(x) - f(0) - \frac{1}{m} A(x) \right| \le \frac{12\delta}{|m|}$$

for all $x \in \mathbb{R}$.

Proof. First, we replace y by $y + m$ in (3.1):

$$|f(xy + mx) + mf(x + y + m) - f(xy + 2mx) - f(my + m^2)| \le \delta. \tag{3.2}$$

By adding (3.1) and (3.2), we get

$$\begin{aligned} &| f(xy) + mf(x+y) - f(my) + mf(x+y+m) - f(xy+2mx) - f(my+m^2) | \\ &\le | f(xy) + mf(x+y) - f(xy+mx) - f(my) | + | f(xy+mx) + mf(x+y+m \\ &f(xy+2mx) - f(my+m^2) | \le 2\delta . \end{aligned}$$

Replacing x by $\frac{x}{2}$ and y by $2y$ in the above inequality, we obtain

$$\begin{aligned} &| f(xy) + mf(\frac{x}{2} + 2y) - f(2my) + mf(\frac{x}{2} + 2y + m) \\ &- f(xy+mx) - f(2my+m^2) | \le 2\delta . \end{aligned} \tag{3.3}$$

From (3.1) and (3.3), we get

$$\begin{aligned} &| mf(\frac{x}{2} + 2y) - f(2my) + mf(\frac{x}{2} + 2y + m) - f(2my+m^2) - mf(x+y) + f(my) | \\ &\le | -f(xy) - mf(x+y) + f(xy+mx) + f(my) | \\ &+ | f(xy) + mf(\frac{x}{2} + 2y) - f(2my) + mf(\frac{x}{2} + 2y + m) - f(xy+mx) - f(2my+m^2) | \\ &\le 3\delta . \end{aligned}$$

Replacing x by $x - y$ in the above inequality, we have

$$\left| mf(\frac{x}{2} + \frac{3y}{2}) + mf(\frac{x}{2} + \frac{3y}{2} + m) - mf(x) - f(2my) - f(2my+m^2) + f(my) \right| \le 3\delta .$$

Letting y by $\frac{y}{3}$, we obtain

$$\left| mf(\frac{x}{2}+\frac{y}{2}) + mf(\frac{x}{2}+\frac{y}{2}+m) - mf(x) - f(\frac{2my}{3}) - f(\frac{2my}{3}+m^2) + f(\frac{my}{3}) \right| \le 3\delta . \tag{3.4}$$

Next, we let $g, h : \mathbb{R} \to \mathbb{R}$ define by

$$\left.\begin{aligned} g(x) &= f(\frac{2mx}{3}) + f(\frac{2mx}{3}+m) - f(\frac{mx}{3}) \\ h(x) &= mf(\frac{x}{2}) + mf(\frac{x}{2}+m). \end{aligned}\right\} \tag{3.5}$$

From (3.4) and (3.5), we get

$$| h(x+y) - mf(x) - g(y) | \le 3\delta . \tag{3.6}$$

Replacing y by 0 in (3.6), we obtain

$$| h(x) - mf(x) - g(0) | \le 3\delta . \tag{3.7}$$

Similarly, we replace x by 0 in (3.6), we have that

$$| h(y) - mf(0) - g(y) | \le 3\delta . \tag{3.8}$$

Next, we define

$$H(x) = h(x) - mf(0) - g(0) . \tag{3.9}$$

By using (3.6), (3.7), (3.8) and (3.9), we have

$$\begin{aligned} | H(x+y) - H(x) - H(y) | &= | h(x+y) - h(x) - h(y) + mf(0) + g(0) | \\ &\le | h(x+y) - mf(x) - g(y) | + | mf(x) - h(x) + g(0) | + | g(y) - h(y) + mf(0) | \\ &\le 9\delta . \end{aligned} \tag{3.10}$$

Now using Hyers theorem [2], we get that

$$| H(x) - A(x) | \le 9\delta , \tag{3.11}$$

where $A : \mathbb{R} \to \mathbb{R}$ is a unhคตe additive function such that $A(x) = \lim_{n\to\infty} \frac{H(2^n x)}{2^n}$. By (3.7), (3.9) and (3.10), we have

$$\begin{aligned} | mf(x) - mf(0) - A(x) | &\le | -h(x) + mf(x) + g(0) | + | h(x) - mf(0) - g(0) - A(x) | \\ &\le 12\delta . \end{aligned}$$

Thus, we obtain

$$| f(x) - f(0) - \frac{1}{m} A(x) | \leq \frac{12\delta}{|m|}$$

for all $x, y \in \mathbb{R}$

Theorem 3.2 *For a fixed* $m, \delta \in \mathbb{R}$ *with* $m \neq 0$ *and* $\delta > 0$. *If the function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the inequality*

$$| f(xy) + mf(x+y) - f(xy+mx) - mf(y) | \leq \delta , \tag{3.12}$$

then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ *such that*

$$| f(x) - f(0) - A(x) | \leq \frac{12\delta}{|m|}$$

for all $m \in \mathbb{R}$.

Proof. First, we substitute y by $y + m$ in (3.12):

$$| f(xy+mx) + mf(x+y+m) - f(xy+2mx) - mf(y+m) | \leq \delta . \tag{3.13}$$

By adding (3.12) and (3.13), we get

$$\begin{aligned} &| f(xy) + mf(x+y) - mf(y) + mf(x+y+m) - f(xy+2mx) - mf(y+m) | \\ &\quad \leq | f(xy) + mf(x+y) - f(xy+mx) - mf(y) | \\ &\qquad + | f(xy+mx) + mf(x+y+m) - f(xy+2mx) - mf(y+m) | \\ &\quad \leq 2\delta . \end{aligned}$$

Replacing x by $\frac{x}{2}$ and y by $2y$ in the above inequality, we get

$$| f(xy) + mf(\frac{x}{2} + 2y) - mf(2y) + mf(\frac{x}{2} + 2y + m) - f(xy+mx) - mf(2y+m) | \leq 2\delta . \tag{3.14}$$

From (3.12) and (3.14), we have that

$$
\begin{aligned}
&|mf(\frac{x}{2}+2y)-mf(2y)+mf(\frac{x}{2}+2y+m)-mf(x+y)+mf(y)| \\
&\qquad \le |-f(xy)-mf(x+y)+f(xy+mx)+mf(y)| \\
&\qquad + |f(xy)+mf(\frac{x}{2}+2y)-mf(2y)+mf(\frac{x}{2}+2y+m)-f(xy+mx)-mf(2y+m)| \\
&\qquad 3\delta .
\end{aligned}
$$

So we get

$$|f(\frac{x}{2}+2y)+f(\frac{x}{2}+2y+m)-f(x+y)-f(2y)-f(2y+m)+f(y)| \le \frac{3\delta}{|m|}.$$

From the last inequality, we replace x by $x-y$ to get

$$|f(\frac{x}{2}+\frac{3y}{2})+f(\frac{x}{2}+\frac{3y}{2}+m)-f(x)-f(2y)-f(2y+m)+f(y)| \le \frac{3\delta}{|m|}.$$

And then substitute y by $\frac{y}{3}$:

$$\left| f(\frac{x}{2}+\frac{y}{2})+f(\frac{x}{2}+\frac{y}{2}+m)-f(x)-f(\frac{2y}{3})-f(\frac{2y}{3}+m)+f(\frac{y}{3}) \right| \le \frac{3\delta}{|m|}. \tag{3.15}$$

Next, we define the function $g,h:\mathbb{R}\to\mathbb{R}$ by

$$\left.\begin{aligned}
g(x) &= f(\frac{2x}{3})+f(\frac{2x}{3}+m)-f(\frac{x}{3}) \\
h(x) &= f(\frac{x}{2})+f(\frac{x}{2}+m).
\end{aligned}\right\} \tag{3.16}$$

From (3.15) and (3.16), we get

$$|h(x+y)-f(x)-g(y)| \le \frac{3\delta}{|m|}. \tag{3.17}$$

From (3.17), we substitute y by 0 and x by 0 respectively to get

$$|h(x)-f(x)-g(0)| \le \frac{3\delta}{|m|} \tag{3.18}$$

and

$$|h(y)-f(0)-g(y)| \le \frac{3\delta}{|m|}. \tag{3.19}$$

Next, we define

$$H(x)=h(x)-f(0)-g(0) \tag{3.20}$$

Using (3.17), (3.18), (3.19) and (3.20), we obtain

$$\begin{aligned}|H(x+y)-H(x)-H(y)| &= |h(x+y)-h(x)-h(y)+f(0)+g(0)| \\ &\le |h(x+y)-f(x)-g(y)|+|f(x)-h(x)+g(0)|+|g(y)-h(y)+f(0)| \\ &\le \frac{9\delta}{|m|}.\end{aligned}$$

By Hyers theorem [2], we obtain

$$|H(x)-A(x)| \le \frac{9\delta}{|m|} \tag{3.21}$$

where $A:\mathbb{R}\to\mathbb{R}$ is a unique additive function such that $A(x)=\lim_{n\to\infty}\frac{H(2^n x)}{2^n}$. Now using (3.18), (3.20) and (3.21), we obtain

$$\begin{aligned}|f(x)-f(0)-A(x)| &\le |f(x)+g(0)-h(x)|+|h(x)-f(0)-g(0)-A(x)| \\ &\le |h(x)-f(x)-g(0)|+|H(x)-A(x)| \\ &\le \frac{12\delta}{|m|}\end{aligned}$$

for all $x\in\mathbb{R}$.

From Theorem 3.1 and Theorem 3.2, if we let m = 1, then we obtain the result of Jung and Sahoo [2]:

Corollary 3.3 *If the function* $f:\mathbb{R}\to\mathbb{R}$ *satisfies the inequality*

$$|f(xy)+f(x+y)-f(xy+x)-f(y)| \le \delta$$

for all $x, y\in\mathbb{R}$, *then there exists a unique additive function* $A:\mathbb{R}\to\mathbb{R}$ *such that*

$$|f(x)-f(0)-A(x)| \le 12\delta$$

for all $x\in\mathbb{R}$.

References

1. Kannappan, Pl. 2009. Functional Equations and Inequalities with Applications. London. Springer. p. 295-298.
2. Sahoo, P. K., and Kannappan, Pl. 2011. Introduction to Functional Equations. Boca Raton. CRC Press-Taylor and Francis Group. p. 295-299, p. 381-384.
3. Davison, T. M. K. 1980. Problem. *Aequationes Mathemsticae* 20: 306.
4. Girgensohn, R., and Lajkó, K. 2000. A Functional Equation of Davison and its Generalization. *Aequationes Mathematicae* 60: 219-224.

ได้รับบทความวันที่ 15 พฤศจิกายน 2556
ยอมรับตีพิมพ์วันที่ 24 ธันวาคม 2556