บทความวิจัย

ควอซีนอร์มอลโหมดของหลุมดำไรส์เนอร์-นอร์ดสเตริมที่มีค่าความโค้งส่วนย่อย $k = -1$, 0, และ 1 ในปริภูมิเวลาแอนไทเดอซิเตอร์ 5 มิติ

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บทคัดย่อ

งานวิจัยนี้เป็นการคำนวณเชิงวิเคราะห์หาควอซีนอร์มอลโหมดของหลุมดำไรส์เนอร์-นอร์ดสเตริม ในปริภูมิเวลา แอนไทเดอซิเตอร์ 5 มิติ หลุมดำนี้ถูกรบกวนด้วยสนามสเกลาร์ที่มีมวลและประจุ โดยประจุนี้ ได้เข้าคู่กับสนามแมกซ์เวลล์ของหลุมดำ และได้ทำการเปลี่ยนค่าความโค้งส่วนย่อยเป็น $k = -1$, 0 และ 1 ผลงานวิจัยที่ได้มีลักษณะเช่นเดียวกับกรณีใน 4 มิติ [5, 6] ในงานวิจัยนี้ยังได้ทำการประมาณพจน์ที่ มีค่าน้อย ณ ที่บริเวณไกลๆ เพื่อตรวจข้อผิดพลาดจากวิธีการประมาณที่ใช้ในงานวิจัย โดยพบว่ายิ่ง ควอซีนอร์มอลโหมดและความถี่ที่มีค่า n สูงเท่าใด ความผิดพลาดจากการประมาณก็จะน้อยลงด้วย

คำสำคัญ: ควอซีนอร์มอลโหมด ความถี่ควอซีนอร์ หลุมดำไรน์เนอร์-นอร์ดสเตริม ปริภูมิเวลาแอน-
ไทเดอซิเตอร์ การแปลงเฟสของหลุมดำ

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Quasinormal Modes of the Reissner-Nordstrom Black Holes with the Sectional Curvature, $k = -1$, 0 and 1, in the 5-Dimensional Anti de Sitter Spacetime

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ABSTRACT

Quasinormal modes of the Reissner-Nordstrom black hole in 5-dimensional AdS spacetimes are analytically calculated. The black holes are perturbed by a charged and massive scalar field. The scalar field charge is coupled to the Maxwell field from the black holes. We vary the sectional curvature as $k = -1$, 0 and 1. The results are similar to those in [5, 6], for 4-dimensional cases. We also approximate some small-value terms at the infinity to check the error from the analytical approximation. The higher value number n of quasinormal modes and their frequencies, the leaser error-value numbers are found.

Keywords: quasinormal modes, quasinorma frequencies, Reissner-Nordstrom black hole, anti de Sitter spacetime, black hole phase transition

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Introduction

Black holes recently have in become an intense field of research. Thanks to the correspondence between the Anti de Sitter spacetime and conformal field theory(AdS/CFT) [1]. Black holes are the regions that the gravitational force is so strong that even light could not escape. Black holes evolve from massive stars with their masses after the supernova 1.4 bigger than the sun mass. These massive stars after the supernova cannot withstand their own gravitational attraction and collapse within. The strong gravitational force causes the highly curved spacetimes in small regions. The surface that even light cannot escape is called the event horizon. All the particles or waves that enter into the black holes are smashed into microscopic particles and flow to the singularity inside. To describe the black hole phenomena one would need a theory that works both in the curved spacetimes and in the microscopic region (quantum theory).

The AdS/CFT correspondence has stated that the general relativity in $n+1$ dimensional AdS spacetime has the same physics as that in the conformal field theory on the n-dimensional surface of the system [1]. The correspondence has offered a chance to study a quantum theory on n dimensions through the $n+1$ dimensional AdS spacetime [2].

Perturbation of black holes in many literatures, e.g. [3], has revealed their phase transition from one kind of a black hole to another kind by changing some parameters. From AdS/CFT, the black hole phase transition is corresponding to the quantum phase transition in many quantum systems, [4].

In [5], the 4-dimensional AdS Reissner-Nordstrom black holes, perturbed by a charged and massive scalar field, are studied numerically, whereas the sectional curvature is zero (k=0). The scalar field charge is coupled to the Maxwell field from the black hole. This study has showed the sign of the black hole phase transition.

In [6], quasinormal modes of the black holes in [5] are analytically calculated and compared, where its results are in agreement with those in [5]. Quasinormal modes are the wave solutions to the Einstein's equations, which satisfy the boundary conditions, i.e. only ingoing wave at the horizon and a finite wave at the infinity (for AdS spacetimes). In [7], the analytical methods in [6] has been taken to calculate the quasinormal modes of the 5 dimensional black holes similar to [5, 6] (k=0). In this work, we continue to calculate quasinormal modes but for the sectional curvature is −1, 0 and 1 (k=−1, 0, 1). We also approximate the error terms that have been thrown away when perform the analytical calculation.

Methods

We start from the Lagrangian of this system [5]

$$\text{Lagrangian} = R + \frac{6}{L^2} - \frac{1}{4}F^2 - \left|\partial_\mu\psi - iqA_\mu\psi\right|^2 - m^2\left|\psi\right|^2 \tag{1}$$

Where R is the scalar curvature. L is the Anti de Sitter radius. $F^2 = F^{\mu\nu}F_{\mu\nu}$ is the Maxwell field in d dimensions. A_μ is the Maxwell potential in d dimensions. In this work we simplify the potential as $A_\mu = (\Phi, 0)$, where Φ is the electric potential due to the black hole charge Q in the coordinate rested relative to the non-rotating black holes. q and m is the charge and mass of the scalar potential ψ. The term $\left|\partial_\mu\psi - iqA_\mu\psi\right|^2 = g^{\mu\nu}(\partial_\mu\psi - iqA_\mu\psi)(\partial_\nu\psi - iqA_\nu\psi)$ presents the electromagnetic interaction between the black hole and the scalar field The metric tensor $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ of AdS spacetime is

$$ds^2 = -fdt^2 + \frac{1}{f}dr^2 + r^2 h_{ij}dx^i dx^j \tag{2}$$

where

$$f(r) = k - \frac{2M}{r^2} + \frac{Q^2}{4r^4} + \frac{r^2}{L^2} \tag{3}$$

$f(r)$ is a function of one variable, radius, r. k is the sectional curvature. In this paper we let $k = -1, 0, 1$, which represent the spacetime geometry symmetry as hyperbolic, flat and spherical. M and Q are the black hole mass and change respectively. h_{ij} is the metric of the angle parts i, $j = 2, 3, 4$ for $d = 5$.

The action in this d-dimensional system is

$$S = \int d^d x \sqrt{-g}\left\{R + \frac{6}{L^2} - \frac{1}{4}F^2 - \left|\partial_\mu\psi - iqA_\mu\psi\right|^2 - m^2\left|\psi\right|^2\right\} \tag{4}$$

$\sqrt{-g}$ is the square root of the metric determent. To the study the perturbation of the scale fiel d ψ in this black hole system, one can vary the action with the scale field ψ to obtain the wave equation of the scale field ψ, i.e.

$$\left[\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right) - m_{eff}^2\right]\psi = 0 \tag{5}$$

m_{eff}^2 is the effective mass.

$$m_{eff}^2 = m^2 + g^{tt}q^2\Phi^2 \tag{6}$$

In this paper the electric potential in d dimensions when $\psi = 0$ is [5], in order to simplify our approximation,

$$\Phi = \sqrt{\frac{d-2}{2(d-3)}}\left(\frac{Q}{r^{d-3}} - \frac{Q}{r_+^{d-3}}\right) \tag{7}$$

For $d=5$, $\Phi(r) = \sqrt{\frac{3}{4}}\left(\frac{Q}{r^2} - \frac{Q}{r_+^2}\right)$. r_+ is the outer horizon of the black hole, which is a solution to eq (3) $f(r) = 0$. There are six solutions, $\pm r_+$ $\pm r_2$ and $\pm r_3$ where r_+ is the largest real positive number among the six solutions.

To solve the wave equation eq(5), one can separate the scalar potential as

$$\psi(t,r,x_i) = e^{-i\omega t} r^{(2-d)/2} R(r) S(x_i) \tag{8}$$

$S(x_i)$ is a harmonic function of the angles x_i, which is satisfied the eigen equation

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\,g^{ij}\frac{\partial}{\partial x^j}\right)S(x_i) = \lambda^2 S(x_i) \tag{9}$$

λ^2 is an eigen value of the harmonic function $S(x_i)$, [5]. For the spherical case $\lambda^2 = l(l+d-3)$, ($l = 0,1,2,3,...$). ω is the frequency of the scalar filed ψ

After substitute eq (8) into eq (5) and apply eq(9) to eliminate the angle variables, one would obtain an ordinary differential equation with the variable radius, r,

$$f\frac{d}{dr}\left(f\frac{dR(r)}{dr}\right) + \left[\omega^2 - V(r)\right]R(r) = 0 \tag{10}$$

where

$$V(r) = \frac{(d-2)(d-4)}{4r^2}f^2 + \frac{\lambda^2}{r^2}f + m_{eff}^2 f + 2\omega q\Phi + \frac{(d-2)}{2r}f\frac{d}{dr}f \tag{11}$$

Let change the variable radius r to z and define new parameters, z_2 and z_3

$$z \equiv \frac{r_+^2}{r^2}, \qquad z_2 = \frac{r_2^2}{r_+^2}, \qquad z_3 = \frac{r_3^2}{r_+^2} \tag{12}$$

The range of variable radius $r_+ < r < \infty$, then the range of the variable changes to $0 < z < 1$. r_+ is satisfied a constrain equation $1 - \frac{8Mr_+^2}{Q^2} + \frac{4r_+^2 k}{Q^2} + \frac{4r_+^6}{L^2 Q^2} = 0$. z_2 and z_3 are equal to

$$z_2, z_3 = \frac{2r_+^4}{Q^2}\left(k + \frac{r_+^2}{Q^2}\right) \pm \frac{2r_+^2}{Q}\sqrt{\frac{r_+^4}{Q^2}\left(k + \frac{r_+^2}{Q^2}\right)^2 + \frac{r_+^2}{L^2}} \tag{13}$$

Let us change the variable r to z in function $f(r) \to f(z)$, $f = \dfrac{Q^2}{4r_+^4 z}(z-1)(z-z_2)(z-z_3)$. At $z = 1$ (at the horizon), $f(z = 1) = 0$, the potential $V(r)$ eq (10) is zero, $V(r = r_+) = 0$ The behavior of the scalar field ψ at the horizon can be approximated from the wave equation eq (10) by defining a new variable as $dr_* = dr/f(r)$. Therefore at the horizon from eq (10), the wave equation is reduced to

$$\frac{d^2 R(r)}{dr_*^2} + \omega^2 R(r) = 0 \tag{14}$$

The solution to equation eq (14) is simply $R(r_*) = e^{\pm i\omega r_*}$, where

$$r_* = \int \frac{dr}{f(r)}$$

$$= \ln\left[\frac{\sqrt{z}-1}{\sqrt{z}+1}\right]^{-\frac{2r_{5+}}{Q^5(1-z_2)(1-z_3)}}\left[\frac{\sqrt{z}-\sqrt{z_2}}{\sqrt{z}+\sqrt{z_2}}\right]^{-\frac{2r_{5+}}{Q^5\sqrt{z_2}(z_2-1)(z_3-z_2)}}\left[\frac{\sqrt{z}-\sqrt{z_3}}{\sqrt{z}+\sqrt{z_3}}\right]^{-\frac{2r_{5+}}{Q^5\sqrt{z_3}(z_3-1)(z_3-z_2)}} \tag{15}$$

and

$$R(r) \approx e^{\pm i\omega r_*}$$

$$= \left[\frac{\sqrt{z}-1}{\sqrt{z}+1}\right]^{\mp\frac{i2\omega r_+^5}{Q^5(1-z_2)(1-z_3)}}\left[\frac{\sqrt{z}-\sqrt{z_2}}{\sqrt{z}+\sqrt{z_2}}\right]^{\mp\frac{2i\omega r_+^5}{Q^5\sqrt{z_2}(z_2-1)(z_3-z_2)}}\left[\frac{\sqrt{z}-\sqrt{z_3}}{\sqrt{z}+\sqrt{z_3}}\right]^{\pm\frac{2\omega r_+^5}{Q^5\sqrt{-z_3}(z_3-1)(z_3-z_2)}} \tag{16}$$

$$\approx (z-1)^{-\frac{i2\omega r_+^5}{Q^5(1-z_2)(1-z_3)}}$$

This means that the behavior of the solution $R(r)$ at the horizon is of the form $R \approx (z-1)^{\alpha_1}$ with

$$\alpha_1^2 = -\frac{4\omega^2 r_+^{10}}{Q^4(1-z_2)^2(1-z_3)^2} \tag{17}$$

The boundary condition at the horizon is the only ingoing into the black hole allowed, corresponding to the negative sign of $\alpha_1 = -\dfrac{i2\omega r_+^5}{Q^2(1-z_2)(1-z_3)}$. The behavior of the solution at z_2 and z_3 are $(z-z_2)^{\alpha_2}$ and $(z-z_3)^{\alpha_3}$ respectively, with

$$\alpha_2^2 = -\frac{4r_+^{10}[\omega + qr_+^2\sqrt{3/4}(z_2-1)/Q]^2}{Q^5 z_2(z_2-1)^2(z_2-z_3)^2} \quad \alpha_3^2 = -\frac{4r_+^{10}[\omega + qr_+^2\sqrt{3/4}(z_3-1)/Q]^2}{Q^5 z_3(z_3-1)^2(z_3-z_2)^2} \quad (18)$$

To solve the wave equation eq(10), let change variable r to z in the wave equation eq(10)

$$
\begin{aligned}
& z^2(z-1)^2(z-z_2)^2(z-z_3)^2\frac{d^2R}{dz^2} + \frac{z}{2}(z-1)^2(z-z_2)^2(z-z_3)^2\frac{dR}{dz} \\
& + z^2(z-1)(z-z_2)^2(z-z_3)^2\frac{dR}{dz} + z^2(z-1)^2(z-z_2)(z-z_3)^2\frac{dR}{dz} \\
& + z^2(z-1)^2(z-z_2)^2(z-z_3)\frac{dR}{dz} - \frac{3}{4^2}(z-1)^2(z-z_2)^2(z-z_3)^2 R \\
& + \frac{3}{4}(z-1)(z-z_2)(z-z_3)\left\{ \begin{array}{l} -(z-1)(z-z_2)(z-z_3) + z(z-z_2)(z-z_3) \\ + z(z-1)(z-z_3) + z(z-1)(z-z_2) \end{array} \right\}R \\
& - \frac{\lambda^2 r_+^4}{Q^2}z(z-1)(z-z_2)(z-z_3)R + \frac{4r_+^{10}}{Q^4}[\omega+q\Phi]^2 zR \\
& - \frac{m^2 r_+^6}{Q^2}(z-1)(z-z_2)(z-z_3)R = 0
\end{aligned}
\tag{19}
$$

And let us write down the solution to eq (19) form as

$$R(z) = z^{\alpha_0}(z-1)^{\alpha_1}(z-z_2)^{\alpha_{21}}(z-z_3)^{\alpha_3}F(z) \tag{20}$$

where

$$\alpha_0 = \frac{1}{4} + \sqrt{1 + \frac{m^2 L^2}{4}} \tag{21}$$

After substitute the solution R(z) from eq (20) into eq (19) and divide the equation by the term $z^{\alpha_0+1}(z-1)^{\alpha_1+1}(z-z_2)^{\alpha_2+1}(z-z_3)^{\alpha_3+1}$ and after some algebra with the property of $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, one would obtain

$$z(z-1)(z-z_2)(z-z_3)\frac{d^2F}{dz^2} + (\frac{1}{2}+2\alpha_0)(z-1)(z-z_2)(z-z_3)\frac{dF}{dz}$$

$$+ (1+2\alpha_1)z(z-z_2)(z-z_3)\frac{dF}{dz} + (1+2\alpha_2)z(z-1)(z-z_3)\frac{dF}{dz} \qquad (22)$$

$$+ (1+2\alpha_3)z(z-1)(z-z_2)\frac{dF}{dz} + J(z)F = 0$$

where

$$J(z) = (\frac{3}{4}+\alpha_0)\{(z-z_2)(z-z_3)+(z-1)(z-z_3)+(z-1)(z-z_2)\}$$

$$+ \alpha_1\left\{\frac{1}{2}(z-z_2)(z-z_3)+z(z-z_3)+z(z-1)\right\}$$

$$+ \alpha_2\left\{\frac{1}{2}(z-1)(z-z_3)+z(z-z_3)+z(z-1)\right\}$$

$$+ \alpha_3\left\{\frac{1}{2}(z-1)(z-z_2)+z(z-z_2)+z(z-1)\right\}$$

$$+ 2\alpha_0\alpha_1(z-z_2)(z-z_3)+2\alpha_0\alpha_2(z-1)(z-z_3)+2\alpha_0\alpha_3(z-1)(z-z_2)$$

$$+ 2\alpha_1\alpha_2 z(z-z_3)+2\alpha_1\alpha_3 z(z-z_2)F+2\alpha_2\alpha_3 z(z-1) \qquad (23)$$

$$- \frac{\lambda^2 r_+^4}{Q^2} + \frac{m^2 r_+^6}{Q^2(-z_2 z_3)}\left[z_2+z_3+z_2 z_3-(1+z_2+z_3)z+z^2\right]$$

$$+ \alpha_1^2\{(z-1)^2+(3-z_2-z_3)(z-1)+2-z_2-z_3+(1-z_2)(1-z_3)\}$$

$$+ \alpha_2^2\{(z-z_2)^2+(3z_2-1-z_3)(z-z_2)+z_2(2z_2-1-z_3)+(z_2-1)(z_2-z_3)\}$$

$$+ \alpha_3^2\{(z-z_3)^2+(3z_3-z_2-z_3)(z-z_3)+z_3(2z_3-1-z_2)+(z_3-1)(z_3-z_2)\}$$

Next divide eq (22) with the term $(Z-Z_2)(Z-Z_3)$

$$z(z-1)\frac{d^2F}{dz^2} + \left\{\begin{array}{l} (\frac{1}{2}+2\alpha_0)(z-1)+(1+2\alpha_1)z \\ \\ + (1+2\alpha_2)\frac{z(z-1)}{z-z_2}+(1+2\alpha_3)\frac{z(z-1)}{z-z_3} \end{array}\right\}\frac{dF}{dz} \qquad (24)$$

$$+ \frac{1}{(z-z_2)(z-z_3)}J(z)F = 0$$

Let us approximate the eq (24) near the horizon, $z = 1$,

$$z(z-1)\frac{d^2F}{dz^2} + \left\{(\frac{1}{2}+2\alpha_0)(z-1)+(1+2\alpha_1)z\right\}\frac{dF}{dz}+J_1F=0 \qquad (25)$$

where

$$J_1 \equiv \frac{J(z=1)}{(1-z_2)(1-z_3)}$$

$$J(z=1)=(\frac{3}{4}+\alpha_0)(z-z_2)(z-z_3)$$

$$+\alpha_1\left\{\frac{1}{2}(z-z_2)(z-z_3)+z(z-z_3)+z(z-z_2)\right\}$$

$$+\alpha_2z(z-z_3)+\alpha_3z(z-z_2)+2\alpha_0\alpha_1(z-z_2)(z-z_3)+2\alpha_1\alpha_2z(z-z_3) \qquad (26)$$

$$+2\alpha_1\alpha_3z(z-z_2)+2\alpha_2\alpha_3z(z-1)-\frac{\lambda^2r_+^4}{Q^2}+\frac{m^2r_+^6}{Q^2(-z_2z_3)}z_2z_3$$

$$+\alpha_1^2\left\{2-z_2-z_3+(1-z_2)(1-z_3)\right\}$$

$$+\alpha_2^2z_2(z_2-z_3)+\alpha_3^2z_3(z_3-z_2)$$

We want to find the solution near the horizon then introduce a new variable $y = 1-z$

$$y(1-y)\frac{d^2F}{dy^2}\left[1+2\alpha_1-(2\alpha_0+2\alpha_1+\frac{3}{2})y\right]\frac{dF}{dy}-J_1F=0 \qquad (27)$$

The two solutions to eq (27) are hypergeometric functions, in this case

$$_2F_1(a,b;1+2\alpha_1;y) \text{ and } (1-z)^{-2\alpha_1}{}_2F_1(a,b;1-2\alpha_1;y) \qquad (28)$$

$$:\alpha_0+\alpha_1+\frac{1}{2}+\sqrt{\left(\alpha_0+\alpha_1+\frac{1}{2}\right)^2+J_1} \quad b=\alpha_0+\alpha_1+\frac{1}{2}-\sqrt{\left(\alpha_0+\alpha_1+\frac{1}{2}\right)^2+J_1}$$

At $Z = 1$, the function $R(z)$ is approximated as

$$R(z)\approx(z-1)^{\alpha_1} \text{ and } R(z)\approx(z-1)^{-\alpha_1}$$

However from the boundary condition at the horizon, we choose $(z-1)^{-\alpha_1}$ and the solution is

$$R(y) = y^{\alpha_1} {}_2F_1(a,b;1+2\alpha_1;y) \tag{29}$$

The other boundary condition is the solution has to be finite at the far away zone, i.e. r $\rightarrow \infty$

$z = \dfrac{r_+^2}{r^2} \rightarrow 0 \; z = 1 - y = \dfrac{r_+^2}{r^2}$. From the property of hypergeometric function, the function

can be transformed from $y = 0$ to $y = 1$ or $z = 0$

$$
\begin{aligned}
R(y) &= z^{\alpha_0} {}_2F_1(a,b;1+2\alpha_1;y) \\
&= z^{\alpha_0} \frac{\Gamma(1+2\alpha_1)\Gamma(1/2-2\alpha_0)}{\Gamma(1+2\alpha_1-a)\Gamma(1+2\alpha_1-b)} {}_2F_1(a,b;\frac{1}{2}+2\alpha_0;z) \\
&+ z^{1/2-\alpha_0} \frac{\Gamma(1+2\alpha_1)\Gamma(-1/2+2\alpha_0)}{\Gamma(a)\Gamma(b)} {}_2F_1(1+2\alpha_1-a,1+2\alpha_1-b;\frac{3}{2}-2\alpha_0;z)
\end{aligned}
\tag{30}
$$

where $\qquad \alpha_0 = \dfrac{1}{4} + \sqrt{1+\dfrac{m^2L^2}{4}}$ and $\dfrac{1}{2} - \alpha_0 = \dfrac{1}{4} - \sqrt{1+\dfrac{m^2L^2}{4}}$

as z $\rightarrow$ 0 one would obtain and $z^{\alpha_0} \rightarrow 0$. The second term in eq (30) is diverged. To eliminate this divergence one can set the parameter $a = -n$ or/and $b = -n$, where n is an integer number ($n=0,1,2,3,..$). therefore

$$\Gamma(a=-n) \rightarrow \infty \text{ or/and } \Gamma(b=-n) \rightarrow \infty \tag{31}$$

or $$n^2 + 2n(\alpha_0 + \alpha_1 + \frac{1}{4})\alpha_0 + J_1 = 0 \tag{32}$$

The frequency ω can be solved from eq (32) by using, the parameters α_1, α_2 and α_3. from eq (17) and eq (18). One can rewrite eq (32) in a form of $a'\omega^2 + c' = 0$, where

$$a' = \frac{4r_+^{10}}{Q^4}\left\{\frac{2}{\sqrt{z_2}\,(z_2-1)^2(z_2-z_3)} - \frac{2i}{\sqrt{-z_2}\,(1-z_3)^2(z_2-z_3)} - \frac{1}{(1-z_2)(1-z_3)}\right\}$$

$$b' = -\frac{i2r_+^5}{Q^2}\left[-2\alpha_0 - \frac{1}{2} - \frac{1}{1-z_2} - \frac{1}{1-z_3} - 2n\right]$$

$$-\frac{i2r_+^5}{Q^2(z_2-z_3)}\left[\frac{i(1-z_3)}{\sqrt{z_2}\,(z_2-1)} + \frac{1-z_2}{\sqrt{-z_2}\,(1-z_3)}\right]$$

$$+\frac{2\sqrt{3}r_+^8 q}{Q^3}\left\{-\frac{1}{\sqrt{z_2}\,(z_2-1)(z_2-z_3)} + \frac{i}{\sqrt{-z_2}\,(1-z_3)(z_2-z_3)} - \frac{1}{(z_2-1)(1-z_3)}\right\}$$

$$c' = \left[n^2 + \frac{n}{2} + 2n\alpha_0 + \frac{3}{4}\right](1-z_2)(1-z_3)$$

$$+\frac{i\sqrt{3}r_+^3 q}{Q}\left[-\frac{i(1-z_3)}{\sqrt{z_2}\,(z_2-z_3)} + \frac{1-z_2}{\sqrt{-z_2}\,(z_2-z_3)}\right] - \frac{\lambda^2 r_+^2}{Q} - \frac{m^2 r_+^6}{Q^2} \tag{33}$$

Therefore the frequency is

$$\omega = -\frac{b'}{2a'} \pm \frac{1}{2a'}\sqrt{b'^2 - 4a'c'} \tag{34}$$

An example for the parameters $k = 0$, $\lambda = 0$, $L = 1.1$, $r_+ = 1$, $Q = 1$, $m^2 L^2 = 4$, $q = 0$, $n = 0$ the frequencies are

$$\omega = 4.53 - 4.55i \text{ and } \omega = -0.46 - 2.66i \tag{35}$$

From the boundary conditions, i.e. only the ingoing wave at the horizon and the non-diverged wave in the far away zone, this allows only the frequency in (35), $\omega = 4.53 - 4.55i$.

Results

The frequencies of the various parameters are plotted, e.g., $k = 0$, $q = 0$, $q = 1$, $n = 0,1,2,..., 10$ $\lambda = 0$, $L = 1.1$ $r_+ = 1$, $Q = 1$, $m^2 L^2 = 4$
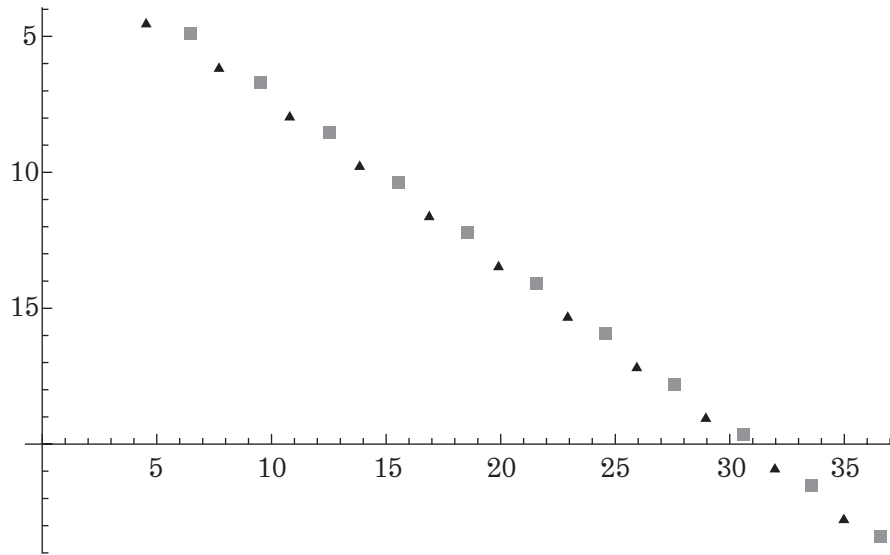
Figure 1 the real and imaginary parts of the frequencies are plotted on horizontal and vertical axes respectively. ▲ and ■ represent $q = 0$ and $q = 1$ respectively. Each value of q represents n=0 to 10 values

From figure 1 both cases, $q = 0$ and $q = 1$ the differences of the frequencies between each n are equal spacing, i.e., $\Delta\text{Re}(\omega) = 3.00$ and $\Delta\text{Im}(\omega) = 1.87$

The frequencies of the various parameters are plotted, e.g., $k = -1$, $q = 0$, $q = 1$, $n = 0,1,2,..., 10$, $\lambda = 0$, $L = 1.1$ $r_+ = 1$, $Q = 1$, $m^2L^2 = 4$
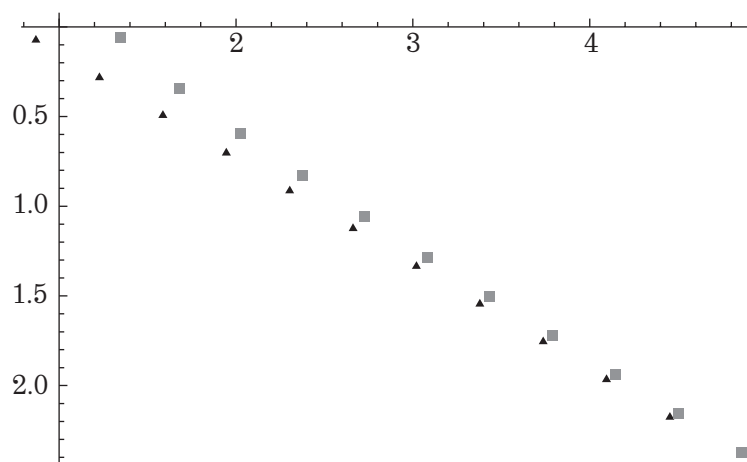

Figure 2 the real and imaginary parts of the frequencies are plotted on horizontal and vertical axes respectively. ▲ and ■ represent $q = 0$ and $q = 1$ respectively. Each value of q represents $n = 0$ to 10 values

From figure 1 both cases, $q = 0$ and $q = 1$ the differences of the frequencies between each n are equal spacing, i.e., $\Delta\mathrm{Re}(\omega) = 0.36$ and $\Delta\mathrm{Im}(\omega) = 0.21$

The frequencies of the various parameters are plotted, e.g., $k = 1$, $q = 0$, $q = 1$, $n = 0,1,2,..., 10$, $\lambda = 0$, $L = 1.1$ $r_+ = 1$, $Q = 1$, $m^2L^2 = 4$
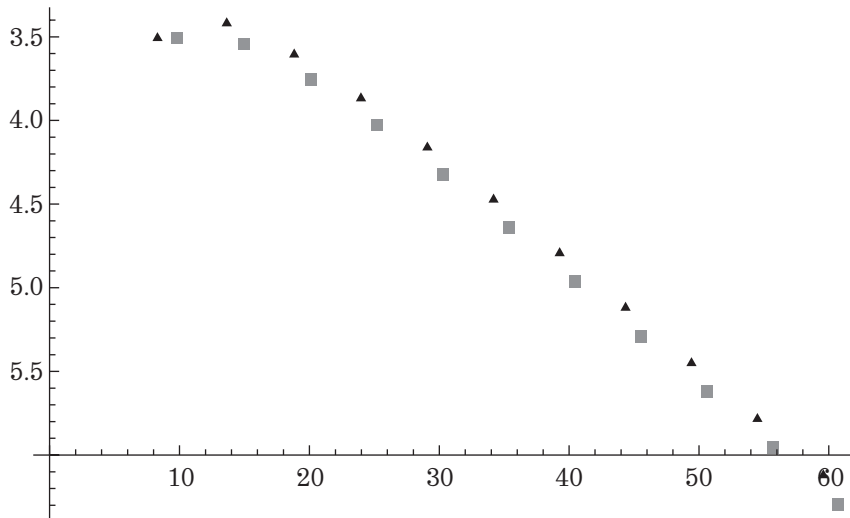

Figure 3 the real and imaginary parts of the frequencies are plotted on horizontal and vertical axes respectively. ▲ and ■ represent $q = 0$ and $q = 1$ respectively. Each value of q represents $n = 0$ to 10 values

From figure 1 both cases, $q = 0$ and $q = 1$ the differences of the frequencies between each n are equal spacing, i.e., $\Delta\mathrm{Re}(\omega) = 5.06$ and $\Delta\mathrm{Im}(\omega) = 0.34$

Eq(27) is an approximation from eq(22). The term that have been thrown away is

$$z(1-z)\left[\frac{(1+2\alpha_2)}{z-z_2} + \frac{(1+2\alpha_3)}{z-z_3}\right]\left[\prod_{i=1}^{n}\frac{(1/2-2\alpha_0-i)}{(1+2\alpha_1+i)}\right]\left[-n^2 - 2(1/4+\alpha_0+\alpha_1)n\right]$$
$$\times\,_2F_1(-n+1,3/2+2\alpha_0+2\alpha_1;3/2+2\alpha_0;z) \tag{36}$$
$$+\{J(z)-J(z=0)\}\,_2F_1(-n,1/2+2\alpha_0+2\alpha_1;1/2+2\alpha_0;z)$$

For $k = 0$, $q = 0$, $\lambda = 0$, $L = 1.1$ $r_+ = 1$, $Q = 1$, $m^2L^2 = 4$ the error is approximated $n = 1$ to 10 as

Table 1 the error from eq(36) at the far away zone, $z = 0$, is showed for the parameters $k = 0$, $q = 0$, $\lambda = 0$, $L = 1.1$ $r_+ = 1$, $Q = 1$, $m^2L^2 = 4$, and the integer n runs from 1 to 10

n	Error from eq (36)	n	Error from eq (36)
1	2.45-i2.40	6	0.05+i0.20
2	-0.34-i2.25	7	0.01-i0.1
3	-1.00-i0.43	8	-0.0014-i0.0038
4	-0.35+i0.26	9	-0.001-i0.00007
5	0.02+i0.15	10	-0.0001+i0.0002

Conclusion and Discussion

Our work both the quasinormal modes and their frequencies are similar to those in [5, 6]. That is a discrete set of frequencies which increases proportionally with the integer number n. In this work we also found that the error of the analytic approximation becomes more reliable when the higher value of n. One can further reduce such the error by performing perturbation where taking our approximated method as the zero-order perturbation and going on collecting all those terms in eq (36) as the first-order perturbation and so on. The quasinormal modes and their frequencies present the states and energy levels of the perturbing wave or particles in this case, the scalar field ψ in which the black holes allow to exist.

Acknowledgements

Reference

1. Maldacena, J. M. 1998. The Large N Limit of Superconformal Field Theories and Supergravity. *Advances in Theoretical and Mathematical Physics* 2: 231-252.

2. Kovtun, K. P., and Starinets, J. M. 2005. Quasinormal Modes and Holography. *Physical Review* D72: 086009.

3. Martinez, C. and Montecinos, A. 2010. Phase Transition in Charged Topological Black Holes Dressed with Scalar Hair. *Physical Review* D82: 127501

4. Steven, G. 2008. Breaking an Abelian Gauge Symmetry near a Black Hole Horizon. *Physical Review* D83: 065034.

5. He, X., Wang, B., Rong-Gen, C., and Chi-Yong, L. 2010. Signature of the Black Hole Phase Transition in Quasinormal Modes. *Physics Letters* B688: 230-236.

6. อรธิรา ฉาบแก้ว. 2555. ควอซีนอร์มอลโหมดของหลุมดำไรส์เนอร์-นอร์ดสเตริม แอนไท เดอ ซิตเตอร์ 4 มิติ ในสนามแม่เหล็กไฟฟ้า ที่มีความโค้งส่วนย่อยเป็นศูนย์. ปริญญานิพนธ์ วท.ม. (ฟิสิกส์). กรุงเทพฯ. บัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ.

7. Jarassriwilai, J., and Musiri, S. 2556. Quasinormal Modes of the Ressner-Nordstrom Black Holes with the Maxwell Field Background in the 5-Dimensional Anti de Sitter Spacetime. การประชุมวิชาการระดับชาติ "ศรีนครินทรวิโรฒวิชาการ" ครั้งที่ 7. 1-2 เมษายน พ.ศ. 2556. กรุงเทพฯ. สถาบันยุทธศาสตร์ทางปัญญาและวิจัย มหาวิทยาลัยศรีนครินทรวิโรฒ. หน้า 655-666.