*บทความวิชาการ*

# ความสมมูลของอสมการแปรผันและรูปแบบไฟไนท์เอลิเมนต์ของปัญหาสิ่งกีดขวางสำหรับเชือกยืดหยุ่น

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## บทคัดย่อ

การศึกษาความสมมูลของอสมการแปรผันและการสร้างรูปแบบไฟไนท์เอลิเมนต์ของปัญหาสิ่งกีดขวางในหนึ่งมิตินี้ได้ทำการศึกษาผ่านปัญหาเส้นลวดไร้น้ำหนักซึ่งถูกแทนที่ด้วยวัตถุชิ้นหนึ่งโดยปัญหาดังกล่าวเป็นปัญหาทางกายภาพอย่างง่ายที่นำไปสู่แนวคิดเชิงคณิตศาสตร์ของการสร้างอสมการแปรผัน เริ่มจากการพิจารณาปัญหาทางกายภาพเพื่อสร้างสมการปัญหาแล้วจะได้ปัญหาอสมการแปรผันจากการพิจารณาปัญหาค่าน้อยสุด ในที่สุด รูปแบบไฟไนท์เอลิเมนท์โดยใช้ฟังก์ชันฐานหลักแบบเชิงเส้นจะถูกสร้างขึ้นภายในปริภูมิไฟไนท์เอลิเมนท์

**คำสำคัญ:** เชือกยืดหยุ่น รูปแบบไฟไนท์เอลิเมนท์ ปัญหาสิ่งกีดขวาง

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# The Equivalence of Variational Inequality and Finite Element Formulation of the Obstacle Problem for a String

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## ABSTRACT

The equivalence of variational inequality and finite element formulation of one dimensional obstacle problem is observed thru the weightless elastic string which is displaced by a body. This is a simple physical problem which will lead to the mathematical idea of a variational inequality formulation. The physical problem is firstly introduced in order to formulate a complementary problem. The variational inequality is then obtained by considering the minimization problem of the energy involving to elastic deformation. Finally, the finite element approximation with linear basis function to variational inequality over the finite element space is presented.

**Keywords:** elastic string, finite element formulation, obstacle problem

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## Physical Problem

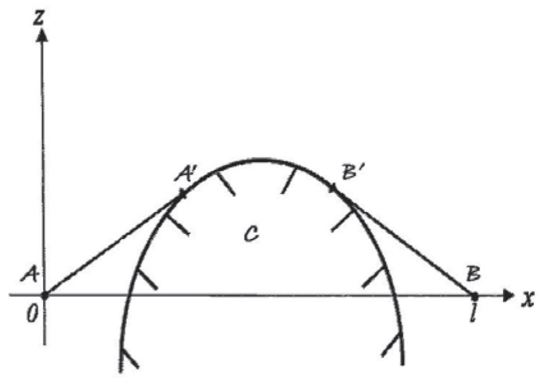


**Figure 1:** Obstacle for a String

Consider the weightless elastic string connected $A$ to $B$, called *string,* which is displaced by a body $C$, called *obstacle.* The shape assumed by the upper part of the boundary of obstacle $C$ and the shape assumed by the string are represented by $\psi(x)$ and $u(x)$, respectively, where, see Figure 1, [3],

$$u''(x) = 0 \text{ in } AA', \ BB'. \tag{1.1}$$

Since the string tends to assume the shape with the minimum possible length, and

$$u = \psi \text{ in } A'B' \tag{1.2}$$

with $u$ and $u'$ being continuous at the unknown boundary $A'B'$, (1.1)-(1.2) are equivalent to

$$\big(u(x)-\psi(x)\big)u''(x) = 0. \tag{1.3}$$

Noting that

$$u(x) \geq \psi(x) \text{ in } AA' \text{ and } BB', \tag{1.4}$$

and, as the string is concave down in $A'B'$,

$$u''(x) \leq 0 \text{ in } A'B'. \tag{1.5}$$

Now, (1.1)-(1.5) can be written on the fixed domain $AB$ as a complementary problem $(PC)$:
**Problem** $(PC)$: Find $u(x)$ such that

$$\big(u(x)-\psi(x)\big)u''(x) = 0 \tag{1.6a}$$

with

$$u(x)-\psi(x) \geq 0, \tag{1.6b}$$

$$u''(x) \leq 0. \tag{1.6c}$$

## Variational Formulation

Since the string $AB$ is being weightless, the only energy, $E$, involved is that due to elastic deformation. Therefore, the minimization problem $(PM)$ can be formulated as follows:
**Problem** $(PM)$ **:** Let $K$ be real Hilbert space defined as

$$K = \left\{ v \in H^1\big([0,l]\big) \mid v(0) = v(l) = 0 \text{ and } v \geq \psi \text{ a.e. in } [0,l] \right\}$$

Find $u(x) \in K$ such that

$$E(u) \leq E(v) \quad \forall v \in K \tag{2.1}$$

where

$$E(v) = \frac{1}{2}\int_0^l \big(v'(x)\big)^2 \, dx .$$

Noting that $H^1\big([0,l]\big)$ denotes the Hilbert space consisting of functions $v$ defined in $[0,l]$ which together with their first derivatives are square-integrable.

Assuming that $u(x) \in K$ is the solution of problem $(PM)$, $\forall v \in K$ and $0 \leq \lambda \leq 1$, then

$$\lambda v(0) + (1-\lambda)u(0) = \lambda v(l) + (1-\lambda)u(l) = 0$$

and

$$\lambda v(x) + (1-\lambda)u(x) \geq \lambda\psi(x) + (1-\lambda)\psi(x) = \psi(x).$$

Therefore, noting the definition of $K$, $\lambda v + (1-\lambda)u \in K$.

Hence, from (2.1) and $u$ is the solution of *(M)*, for small positive $\lambda$,

$$\int_0^l (u')^2 dx \le \int_0^l (\lambda v' + (1-\lambda)u')^2 dx$$

$$= \int_0^l (u')^2 dx + 2\lambda \int_0^l u'(v'-u')dx + O(\lambda^2). \quad (2.2)$$

Dividing (2.2) by $\lambda$ and, then, letting $\lambda \to 0^+$ yields

$$\int_0^l u'(v'-u')dx \ge 0.$$

Now let $H_0^1([0,l])$ be $H^1([0,l])$ with boundary conditions $v(0)=v(l)=0$ denoted by $H_0^1([0,l]) = \{v \in H^1([0,l]) \mid v(0)=v(l)=0\}$ and $a(.,.)$ be a bilinear form expressed as

$$a(u,v) = \int_0^l u'(x)v'(x)dx$$

where $a(.,.) : (H_0^1([0,l]) \times H_0^1([0,l])) \to \mathbb{R}$, such that

1. $a(.,.)$ is symmetric, i. e.,

$$a(u,v) = a(v,u) \qquad \forall u,v \in H_0^1([0,l]),$$

2. $a(.,.)$ is coercive, i. e., there is a constant $\alpha > 0$ such that

$a(v,v) \ge \alpha |v|_1^2 \quad \forall v \in H_0^1$, and $|v|_1 = \left( \int_0^1 \left(v^2 + (v')^2\right) dx \right)^{\frac{1}{2}}$ is a norm for $H_0^1([0,l])$,

3. $a(.,.)$ is continuous, i. e., there is a constant $\gamma > 0$ such that

$$|a(u,v)| \le \gamma |u|_1 |v|_1 \qquad \forall u,v \in H_0^1([0,l]).$$

Therefore, problem *(PM)* implies the variational inequality problem *(PV)*:

**Problem** *(PV)* : Find $u(x) \in K$ such that

$$a(u, v-u) \ge 0 \qquad \forall v \in K.$$

Let us show that the solution $u$ of problem *(PV)* is also a solution of *(PM)*. Suppose that $u$ solves problem *(PV)* then consider, for any $v \in K \subseteq H_0^1([0,l])$,

$$\frac{1}{2}\left[a(v,v)-a(u,u)\right] = \frac{1}{2}\left[a(v,v)+a(u,v)-a(u,v)-a(u,u)\right]$$

$$= \frac{1}{2}\left[a(v,v)+a(u,v-u)-a(u,v)\right]$$

$$\geq \frac{1}{2}\left[a(v,v)-a(u,v)\right]$$

$$= \frac{1}{2}\left[a(v-u,v)-a(v-u,u)+a(v-u,u)\right]$$

$$= \frac{1}{2}\left[a(v-u,v-u)+a(v-u,u)\right]. \tag{2.3}$$

Recalling that $a(.,.)$ is coercive, then, $a(v-u,v-u)\geq\alpha\left|v-u\right|_1^2\geq 0$, for $\alpha>0$, and (2.3) becomes

$$\frac{1}{2}\big(a(v,v)-a(u,u)\big) \geq 0 \quad \text{or} \quad E(v)\geq E(u)$$

which is problem *(PM)* being equivalent to problem *(PV)*. Noting that the existence and uniqueness of problem *(PM)* and *(PV)* are followed by the Lax-Milgram theorem†.

Now letting $u,v\in K$ and multiplying any function $(v-u)\geq 0\in K$ to both sides of (1.6c),

$$u''(v-u) = u''v-u''u \leq 0. \tag{2.4}$$

Integrating (2.4) by parts over 0 to *l*,

$$-(u',v')+(u',u') \leq 0$$

or

$$(u',v')-(u',u') = (u',v'-u')\geq 0$$

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†**Lax-Milgram Theorem:** Given a real Hilbert space, $V(.,.)$, a continuous, coercive bilinear functional $a(.,.)$ on $V \times V$ and a continuous linear functional $F\in V'$, then there exists a unique $u\in V$ such that $a(u,v)=\langle F,v\rangle_{V',V}$, $\forall v\in V$, furthermore, where $\alpha$ is the coercive constant, $\|v\|_V\leq\frac{1}{\alpha}\|F\|_{V'}$.

which is problem *(PV)*. Therefore, the complimentary problem *(PC)* implies the variational problem *(PV)*.

Assuming that $u \in K$ satisfies

$$\int_0^l u'v'dx - \int_0^l u'u'dx \geq 0, \qquad \forall v \in K$$

and $u''$ exists and is continuous, then the both terms on the left-hand side can be integrated by parts. Noting that $u(0) = u(l) = v(0) = v(l) = 0$, thus

$$-\int_0^l u''vdx + \int_0^l u''udx \geq 0 \text{ or } \int_0^l u''(v-u)dx \leq 0. \tag{2.5}$$

Since the continuity of $u''$, (2.5) can be hold only if $u'' \leq 0$. This shows that the variational problem *(PV)* also implies the complimentary problem *(PC)*. Up until now the equivalence of problem *(PC)*, *(PM)* and *(PV)* is possessed and can be written symbolically as $(PC) \Leftrightarrow (PV) \Leftrightarrow (PM)$.

## Finite Element Approximation

Let $K^h$ be the finite element space approximating $K$ and consisting of piecewise linear functions. $K^h$ is approximated by

$$K^h = \left\{ v \mid v = \sum_{j=1}^{M} v_j \varphi_j(x) \in V^h \text{ and } v_j \geq \psi(x_j)\ j = 1, \ldots, M \right\}$$

where we let $V^h$ be the finite element space approximating $H_0^1([0,l])$, $v_j$ be the value at the node $x_j$ and $\varphi_j$ be a basis (or shape) function for $V^h$ defined by, for *i, j* = 1,2,...,*M,*

$$\varphi_j(x_i) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

i.e., $\varphi_j$ is the continuous piecewise linear function that takes the value 1 at node point $x_j$ and the value 0 at other node points.

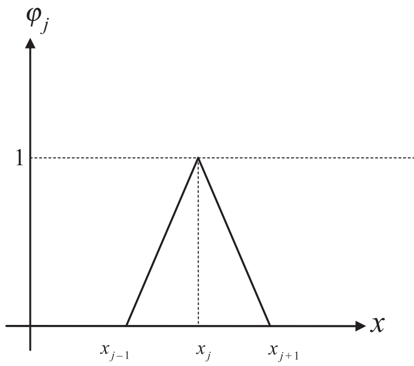


**Figure 2:** Basis Function $\varphi_j$

For a finite element discretization, the variational inequality problem *(PV)* is considered over $K^h$, that is

**Problem** *(FV)* : Find $u^h \in K^h$ such that

$$a(u^h, v^h - u^h) \geq 0 \text{ for all } v^h \in K^h \tag{3.1}$$

which is equivalent to

**Problem** *(FM)* : Find $u^h \in K^h$ such that

$$E\left(u^h\right) \leq E\left(v^h\right) \text{ for all } v^h \in K^h. \tag{3.2}$$

For simplicity, let $0 = x_0 < x_1 < \ldots < x_M < x_{M+1} = l$ be a partition of the interval $[0,l]$ into subintervals $I_j = \left(x_{j-1}, x_j\right)$ of uniform length $h = \dfrac{x_{M+1} - x_0}{M+1}$. Since $u^h, v^h \in K^h$ and their representation are

$$u^h = \sum_{i=1}^{M} u_i \varphi_i(x) \text{ and } v^h = \sum_{i=1}^{M} v_i \varphi_i(x),$$

respectively, where $u_i = u_i(x_i)$, $v_i = v_i(x_i)$ and $x \in [0,l]$, (3.1) becomes

$$0 \leq a(u^h, v^h - u^h) = \sum_{i=1}^{M} \sum_{j=1}^{M} a\left(u_i \varphi_i, \left(v_j - u_j\right) \varphi_j\right)$$

$$= \sum_{i=1}^{M} \sum_{j=1}^{M} u_i a\left(\varphi_i, \varphi_j\right)(v_j - u_j). \tag{3.3}$$

Now, we shall show that *(FV)* implies the complementary problem. For simplicity, we define $\psi_j = \psi(x_j)$ . Fixing $j = k$ and taking

$$v_i = \begin{cases} u_i & \text{if } i \neq k, \\ \psi_k & \text{if } i = k \end{cases}$$

in (3.3) yields

$$\sum_{i=1}^{M} u_i a(\varphi_i, \varphi_k)(\psi_k - u_k) \geq 0 \text{ for all } k. \qquad (3.4)$$

Again, taking

$$v_i = \begin{cases} u_i & \text{if } i \neq k, \\ 2u_k - \psi_k & \text{if } i = k \end{cases}$$

in (3.3) yields

$$\sum_{i=1}^{M} u_i a(\varphi_i, \varphi_k)(u_k - \psi_k) \geq 0 \text{ for all } k. \qquad (3.5)$$

Therefore, by (3.4) and (3.5),

$$\sum_{i=1}^{M} u_i a(\varphi_i, \varphi_k)(u_k - \psi_k) = 0 \text{ for all } k. \qquad (3.6)$$

Consider, again, noting (3.3) and (3.6),

$$0 \leq \sum_{i=1}^{M}\sum_{j=1}^{M} u_i a(\varphi_i, \varphi_j)(v_i - u_j) = \sum_{i=1}^{M}\sum_{j=1}^{M} u_i a(\varphi_i, \varphi_j)(v_i - \psi_j + \psi_j - u_j)$$

$$= \sum_{i=1}^{M}\sum_{j=1}^{M} u_i a(\varphi_i, \varphi_j)(v_j - \psi_j).$$

Now taking, for fixed $k$,

$$v_i = \begin{cases} u_i & \text{if } i \neq k, \\ \psi_j + 1 & \text{if } i = k. \end{cases}$$

Hence

$$\sum_{i=1}^{M} u_i a\left(\varphi_i, \varphi_k\right) \geq 0. \tag{3.7}$$

It follows that problem *(FV)*, with (3.6) and (3.7), implies the complementary problem which can be written in matrix form as, since $u_i \geq \psi_i$ for all $i$,

**Problem** *(FC)* : Find $\overline{u}$ such that

$$\left(A\overline{u}\right)^T \left(\overline{u} - \overline{\psi}\right) = 0 \tag{3.8a}$$

with

$$A\overline{u} \geq 0, \tag{3.8b}$$

$$\overline{u} \geq \overline{\psi}, \tag{3.8c}$$

where $A_{ij} = a\left(\varphi_i, \varphi_j\right)$ is the $M \times M$ symmetric positive definite matrix since $a(.,.)$ is a symmetric coercive bilinear form, and the vectors $\{\overline{u}, \overline{v}, \overline{\psi}\}$ where $\overline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix}$, $\overline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_M \end{bmatrix}$ and $\overline{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_M \end{bmatrix}$ are the nodal values.

Let us show that problem *(FC)* also implies *(FV)*. Consider at $i^{th}$ point. If $u_i > \psi_i$, on noting (3.8b-c), we must have $\left(A\overline{u}\right)_i = 0$, thus

$$\left(A\overline{u}\right)_i \left(v_i - u_i\right) = 0.$$

And, if $u_i = \psi_i$, then

$$A\overline{u} \geq 0 \text{ and also } v_i - u_i = v_i - \psi_i \geq 0.$$

We, hence, obtain

$$\left(A\overline{u}\right)_i \left(v_i - u_i\right) \geq 0.$$

Therefore, it follows that

$$\sum_{i=1}^{M} \sum_{j=1}^{M} u_j A_{ij} (v_j - u_j) = \left(A\overline{u}\right)^T \left(\overline{v} - \overline{u}\right)$$

$$a\left(u^h, v^h - u^h\right) \geq 0.$$

Recently, the equivalence of the variational problem *(FV)* and the complementary problem *(FC)* have been shown. Since the problem *(FV)* is also equivalent to minimisation problem *(FM)*, therefore the three problem *(FV)*, *(FM)* and *(FC)* are equivalents which can be written symbolically as *(FV)* $\Leftrightarrow$ *(FM)* $\Leftrightarrow$ *(FC)*.

To compute $A_{ij}$, observing firstly that $a\left(\varphi_i,\varphi_j\right)=0$ if $\left|i-j\right|>1$ since in this case either $\varphi_i$ or $\varphi_i$ is equal to zero, we thus have for $j = 1,\ldots,M$,

$$
\begin{aligned}
a\left(\varphi_j,\varphi_j\right) &= \int_{x_j-1}^{x_j+1} \varphi_j'\varphi_j'\,dx \\
&= \int_{x_{j-1}}^{x_j} \frac{1}{h^2}dx + \int_{x_j}^{x_{j+1}} \frac{1}{h^2}dx \\
&= \frac{2}{h},
\end{aligned}
$$

and for $j = 2,\ldots,M$,

$$
\begin{aligned}
a\left(\varphi_j,\varphi_{j-1}\right) &= a\left(\varphi_{j-1},\varphi_j\right) \\
&= -\int_{x_{j-1}}^{x_j} \frac{1}{h^2}dx \\
&= -\frac{1}{h}.
\end{aligned} \tag{3.9}
$$

Thus the stiffness matrix $A$ is expressed as

$$
A = \frac{1}{h}\begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & -1 & \ddots & & \vdots \\
0 & -1 & 2 & -1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & -1 & 2 & -1 \\
0 & \cdots & \cdots & 0 & -1 & 2
\end{pmatrix}
$$

which is symmetric and positive definite[‡].

[‡]**Positive Definite Matrix:** A symmetric $M \times M$ matrix $A$ is said to be positive definite if for any non zero vector $x$ then $x^T Ax > 0$.

## Conclusion

To sum up, the equivalence of complementary problem, minimization problem and variational problem of the weightless elastic string with obstacle have been presented in both continuous and discrete space. The possessed finite element formula leads to a system equations with a symmetric, positive definite and sparse stiffness matrix.

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