บทความวิจัย

ขอบเขตล่างที่ชัดเจนของพลังงานสถานะพื้นของสสารที่เป็นไปตามหลักการกีดกันในสองมิติ

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บทคัดย่อ

ขอบเขตล่างที่ชัดเจนของพลังงานสถานะพื้นของสสารที่เป็นกลางประเภทเฟอร์มิออนในสองมิติภายใต้อันตรกิริยาคูลอมบ์ โดยที่ประจุบวกถูกกำหนดให้อยู่กับที่ คือ $E_N > -c_F N$ ในหน่วยไรด์เบิร์กได้จากการพิจารณาขอบเขตล่างของพลังงานจลน์ในรูปยกกำลังอินทิกรัลของ ρ^2 เมื่อ ρ คือ ความหนาแน่นของอนุภาค ยิ่งไปกว่านั้น ยังพบว่าพลังงานที่สถานะพื้นของสสารแปรผกผันกับตัวคูณสปิน

คำสำคัญ: พลังงานสถานะพื้น เสถียรภาพ ขอบเขตล่าง สสารประเภทเฟอร์มิออน

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Rigorous Lower Bound for the Ground State Energy of Matter with the Exclusion Principle in Two Dimensions

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ABSTRACT

The rigorous lower bound for the ground state energy, $E_N > -c_F N$ in Rydberg unit, in two dimensions of neutral matter of fermionic types with Coulomb interactions with fixed positive charges is possessed by considering, in process, lower bound for the kinetic energy as some power of an integral of ρ^2 where ρ is the particle density and, moreover, it is the inverse proportion with the spin multiplicity.

Keywords: ground state energy, stability, lower bound, fermionic matter

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Introduction

A rigorous study of instability and stability of such systems for matter began several years ago. The remarkable work of Dyson and Lenard [2] have been giving rise to the famous N^{α} power laws for the ground state energy. Many simplified derivation by Lieb and Thirring [7-9] with tremendous improvements of the corresponding estimates have been given for the "bosonic matter", matter without the exclusion principle, and "fermionic matter", matter with the exclusion principle.

In the case of bosonic matter in two dimensions, the upper bound of the ground state energy (E_N) have been derived by Muthaporn and Manoukain [12,13], as

$$E_N < -0.0002N^2$$

whereas Shiwongsa, Sirininlakul and Sripirom [16] possessed its exact lower bound,

$$E_N > -4(1-Z_{\max})N^2$$

which are satisfied the estimate for the upper and lower bounds of such matter in two dimensions $E_N < -c_U N^2$ and $E_N > -c_L N^2$ respectively where c_U and c_L are positive constants.

Nevertheless, in order to investigate the nature of matter in two dimensions, there is an important theoretical question:- *will the matter change from a stable phase to an unstable or explosive phase provide that the matter change from bosonic type to fermionic type?* Consequently, the investigation of fermionic matter has been also important to study, especially, for the nature of a matter in two dimensions with the exclusion principle.

In this paper, to answer the above question, we derive a rigorous lower bound for the ground state energy of the system with *N* electrons and *N* motionless positive charges with Columbic interactions with the roles of the spin and statistics theorem. We also show that fermionic matter is stable in two dimensions. The Hamiltonian considered in this paper is

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \sum_{i<j}^{N} e^2 \left|\mathbf{x}_i - \mathbf{x}_j\right|^{-1} + \sum_{i<j}^{k} Z_i Z_j e^2 \left|\mathbf{R}_i - \mathbf{R}_j\right|^{-1} - \sum_{i=1}^{N}\sum_{j=1}^{k} Z_j e^2 \left|\mathbf{x}_i - \mathbf{R}_j\right|^{-1} \quad (1.1)$$

where $\mathbf{x}_i$ and $\mathbf{R}_i$, respectively, denote the position of negative and positive charges, and $\sum_{i=1}^{N} Z_i = N$, $k \geq 2$ with fixed (motionless) positive charges. We note that on setting, $\frac{1}{4\pi\varepsilon_0} = 1$, the third term in the right-hand side of (1.1) will be absent in the expression for H and one would be dealing with an atom. Throughout this paper, we are interested in the case for which $k \neq 1$ relevant to matter.

The Lower Bound of the Kinetic Energy

In order to obtain a lower bound for the ground-state energy of matter, first we have to find a lower bound for the expectation value kinetic energy, T, which is the first term on the right-hand side of (1.1). For simplicity, we consider the fermionic of spin $\frac{1}{2}$ (electron). In multi-particle systems, we also consider N identical fermions in each of mass m and introduce the particle number density in two dimensions as

$$\rho(\mathbf{x}) = N \sum_{\sigma_1, \sigma_2, \ldots, \sigma_N} \int d^2\mathbf{x}_2 \cdots d^2\mathbf{x}_N \left|\psi(\mathbf{x}_1\sigma_1, \mathbf{x}_2\sigma_2, \ldots, \mathbf{x}_N\sigma_N)\right|^2,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_N$ specify spin projection values of each spin multiplicity $\overline{\zeta} = 2s+1$ for a particle of spin s and ψ is an N -fermion anti-symmetric normalized wave function. The total number of particle, N, is obtained from the normalization condition

$$\int d^2\mathbf{x}\, \rho(x) = N$$

and the wave functions $\psi(\mathbf{x}_1\sigma_1, \mathbf{x}_2\sigma_2, \ldots, \mathbf{x}_N\sigma_N)$ are assumed to satisfy the appropriate statistics being anti-symmetric in the exchange of any two electrons which leads to the equivalence of the interchange of the position-spin labeling symbolically as $(\mathbf{x}_i\sigma_j) \Leftrightarrow (\mathbf{x}_j\sigma_i)$.

We now apply the Schwinger inequality [14], for a positive number ξ, the number of eigenvalues (counting degeneracy) of a Hamiltonian $\frac{\mathbf{p}^2}{2m} - g(\mathbf{x})$ for $k \geq 2$ in two dimensions is equal to or less than -ξ, it follows that

$$N_{-\xi}(H_0 - g(\mathbf{x})) \leq \frac{m}{2\pi\hbar^2\xi} \int d^2\mathbf{x} \left[g(\mathbf{x})\right]^2 \tag{2.1}$$

where $g(\mathbf{x})$ is positive function. Recently, an exact functional expression for $N_{-\xi}(H_0 - g(\mathbf{x}))$ derived by Manoukian and Limboonsong [10] is not as upper bound as in (2.1).

For $N_{-\xi}(H_0 - g(\mathbf{x})) < 1$ and any $\delta > 0$, we may choose ξ in (2.1) such that

$$-\xi = -\frac{m}{2\pi\hbar^2}(1+\delta)\int d^2\mathbf{x}\left[g(\mathbf{x})\right]^2, \tag{2.2}$$

so that $N_{-\xi}\left(\frac{\mathbf{p}^2}{2m} - g(\mathbf{x})\right) < 1$, which implies that $N_{-\xi}\left(\frac{\mathbf{p}^2}{2m} - g(\mathbf{x})\right) = 0$ and the right-hand side of (2.2) provides a lower bound to the spectrum of $\frac{\mathbf{p}^2}{2m} - g(\mathbf{x})$ since its spectrum would then be empty for energies are equal or less than -ξ. Therefore, (2.2) gives the following lower bound for the ground-state energy of the Hamiltonian as

$$-\frac{m}{2\pi\hbar^2}(1+\delta)\int d^2\mathbf{x}\left[g(\mathbf{x})\right]^2. \tag{2.3}$$

To obtain the lower bound for kinetic energy, T, of one particle systems, we first consider the one particle with $\int d^2\mathbf{x}\,\rho(\mathbf{x}) = 1$ and define the positive function

$$g(\mathbf{x}) = \frac{\gamma\rho^{\alpha}(\mathbf{x})}{\int d^2\mathbf{x}\,\rho^{\alpha+1}(\mathbf{x})}T \tag{2.4}$$

where γ and α are positive, and $g(\mathbf{x})$ is not the potential energy for any Hamiltonian. This is introduced only in order to be able to obtain a lower bound for T. Substituting (2.4) into $\left\langle \psi \left| \frac{\mathbf{p}^2}{2m} - g(\mathbf{x}) \right| \psi \right\rangle$, we have that

$$\left\langle \psi \left| \frac{\mathbf{p}^2}{2m} - g(\mathbf{x}) \right| \psi \right\rangle = -(\gamma - 1)T. \tag{2.5}$$

Referring to the bound in (2.3) yields

$$\left\langle \psi \left| \frac{\mathbf{p}^2}{2m} - g(\mathbf{x}) \right| \psi \right\rangle \geq -\frac{m}{2\pi\hbar^2}(1+\delta)\int d^2\mathbf{x}\left[g(\mathbf{x})\right]^2. \tag{2.6}$$

Then on noting (2.5) and (2.6), it follows that

$$(\gamma - 1)T \geq \frac{\gamma^2 T^2 m}{2\pi\hbar^2}(1+\delta)\frac{\int d^2\mathbf{x}\,\rho^{2\alpha}(\mathbf{x})}{\left[\int d^2\mathbf{x}\,\rho^{\alpha+1}(\mathbf{x})\right]^2}. \tag{2.7}$$

Choosing $2\alpha = \alpha + 1$ in the right-hand side of (2.7), we hence have $\alpha = 1$ and the inequality becomes

$$T \geq \left(\frac{\gamma - 1}{\gamma^2}\right)\left(\frac{\pi}{(1+\delta)}\right)\left(\frac{2\hbar^2}{m}\right)\left(\int d^2\mathbf{x}\,\rho^2(\mathbf{x})\right)^{\frac{2}{3}}. \tag{2.8}$$

Optimizing (2.8) over γ yields

$$\gamma = 2.$$

Together with $\alpha = 1$ we obtain the positive function $g(\mathbf{x})$ in term of $\rho^2(\mathbf{x})$ as

$$g(\mathbf{x}) = 2T\frac{\rho(\mathbf{x})}{\int d^2(\mathbf{x})\rho^2(\mathbf{x})}.$$

Noting (2.2), for N identical fermions, we obtain

$$\left\langle \psi \left| \sum_{i=1}^{N} g(\mathbf{x}_i) \right| \psi \right\rangle = 2T$$

where $\sum_{i=1}^{N} g(\mathbf{x}_i) = g(\mathbf{x})$.

Considering the operator

$$\sum_{i=1}^{N}\left(\frac{\mathbf{p}_i^2}{2m} - g(\mathbf{x}_i)\right) \tag{2.9}$$

defining a hypothetical Hamiltonian of N non-interacting fermions which, however, interact with the external "potential", $g(\mathbf{x})$ we have

$$\left\langle \psi \left| \sum_{i=1}^{N}\left(\frac{\mathbf{p}_i^2}{2m} - g(\mathbf{x}_i)\right) \right| \psi \right\rangle = -T\,. \tag{2.10}$$

To obtain a lower bound to the lower end of the spectrum of the "Hamiltonian" operator, (2.9), we note that, allowing for multiplicity and spin degeneracy, the N fermions can be placed in the lowest energy levels of the "Hamiltonian" to conform with Pauli's exclusion principle. To define the lowest energy of the Hamiltonian, if N is larger than this number of levels, the remaining free fermions may be chosen to have arbitrary small kinetic energies ($T \rightarrow 0$) and be infinitely separated. That is, in all cases, the Hamiltonian (2.9) is bounded below by $\overline{\zeta}$ times the ground-state energy (2.2). For N identical fermions, it follow from (2.6) that

$$\left\langle \psi \left| \sum_{i=1}^{N} \left(\frac{\mathbf{p}_i^2}{2m} - g(\mathbf{x}_i) \right) \right| \psi \right\rangle \geq -\frac{\bar{\zeta} m}{2\pi\hbar^2}(1+\delta)\int d^2\mathbf{x}\left[g(\mathbf{x})\right]^2 . \tag{2.11}$$

On noting (2.10) and (2.11), we finally have the expectation value of the kinetic energy T as

$$T \geq \frac{\pi\hbar^2}{2\bar{\zeta} m(1+\delta)}\int d^2\mathbf{x}\,\rho^2(\mathbf{x}) \tag{2.12}$$

for any $\delta > 0$ and N identical fermions.

The Exact Bound for Coulomb Potential

We first apply the general bound for Coulomb potential in two dimensions which derived in Shiwongsa, Sirininlakul and Sripirom[16] as

$$\sum_{i<j}^{k} \frac{e^2 A_i A_j}{\left|\mathbf{x}_i - \mathbf{x}_j\right|} = \sum_{j=1}^{N} e^2 A_j \int d^2\mathbf{x} \frac{\rho(\mathbf{x})}{\left|\mathbf{x}_i - \mathbf{x}_j\right|} - \frac{e^2}{2}\int d^2\mathbf{x}' \int d^2\mathbf{x} \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|} - \frac{e^2\lambda_0}{2}\sum_{j=1}^{N} A_j^2 - \frac{\pi e^2}{\lambda_0^2}\int d^2\mathbf{x}\rho^2(\mathbf{x}) \tag{3.1}$$

where $\lambda = \lambda_0$ is a parameter derived by optimizing the Hamiltonian. It is then straightforward to apply (3.1) twice, once to the repulsive potential (electron-electron interaction), the second term in the right-hand side of (1.1). Let $A_i = A_j = 1$ and $k \to N$, we obtain

$$\sum_{i<j}^{k} \frac{e^2}{\left|\mathbf{x}_i - \mathbf{x}_j\right|} = \sum_{j=1}^{N} e^2 \int d^2\mathbf{x} \frac{\rho(\mathbf{x})}{\left|\mathbf{x}_i - \mathbf{x}_j\right|} - \frac{e^2}{2}\int d^2\mathbf{x}' \int d^2\mathbf{x} \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|} - \frac{e^2\lambda_0}{2}\sum_{j=1}^{N} 1 - \frac{\pi e^2}{\lambda_0^2}\int d^2\mathbf{x}\rho^2(\mathbf{x}) \tag{3.2}$$

and, again, to the repulsive potential (nucleus-nucleus interaction), the third term in the right-hand side of (1.1). Let $A_i = Z_i$, $A_j = Z_j$ and $\mathbf{x}_j \to \mathbf{R}_j$, we also have, for $k \geq 2$,

$$\sum_{i<j}^{k} \frac{e^2 Z_i Z_j}{\left|\mathbf{R}_i - \mathbf{R}_j\right|} = \sum_{j=1}^{N} e^2 Z_j \int d^2\mathbf{x} \frac{\rho(\mathbf{x})}{\left|\mathbf{x} - \mathbf{R}_j\right|} - \frac{e^2}{2}\int d^2\mathbf{x}' \int d^2\mathbf{x} \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|} - \frac{e^2\lambda_0}{2}\sum_{j=1}^{N} Z_j^2 - \frac{\pi e^2}{\lambda_0^2}\int d^2\mathbf{x}\rho^2(\mathbf{x}) . \tag{3.3}$$

We substitute (3.2), (3.3), $\sum_{j=1}^{N} 1 = N$ and $\sum_{i=1}^{k} Z_i = N$ where $k \geq 2$, into (1.1) the ground state energy, $\langle \psi | H | \psi \rangle$, is expressed as, for $k \geq 2$,

$$\langle \psi | H | \psi \rangle = T + \left\langle \psi \left| \sum_{j=1}^{N} e^2 \int d^2\mathbf{x} \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_j|} \right| \psi \right\rangle + \left\langle \psi \left| \sum_{j=1}^{k} e^2 Z_j \int d^2\mathbf{x} \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{R}_j|} \right| \psi \right\rangle$$
$$- \left\langle \psi \left| \frac{2\pi e^2}{\lambda_0^2} \int d^2\mathbf{x} \rho^2(\mathbf{x}) \right| \psi \right\rangle - \left\langle \psi \left| \frac{e^2 \lambda_0}{2} \left(N + \sum_{i=1}^{k} Z_i^2 \right) \right| \psi \right\rangle$$
$$- \left\langle \psi \left| e^2 \int d^2\mathbf{x}' \int d^2\mathbf{x} \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right| \psi \right\rangle - \left\langle \psi \left| \sum_{i=1}^{N} \sum_{j=1}^{k} \frac{e^2 Z_j}{|\mathbf{x}_i - \mathbf{R}_j|} \right| \psi \right\rangle. \tag{3.4}$$

Lower Bound for the Ground State Energy

On noting (3.4), we obtain the ground-state energy of N identical fermions as, for $k \geq 2$,

$$\langle \psi | H | \psi \rangle = T - \left\langle \psi \left| \frac{2\pi e^2}{\lambda_0^2} \int d^2\mathbf{x} \rho^2(\mathbf{x}) \right| \psi \right\rangle - \left\langle \psi \left| \frac{e^2 \lambda_0}{2} \left(N + \sum_{i=1}^{k} Z_i^2 \right) \right| \psi \right\rangle. \tag{4.1}$$

Optimizing (4.1) over λ_0 gives

$$\lambda_0 = \left(4\pi \frac{\int d^2\mathbf{x} \rho^2(\mathbf{x})}{N + \sum_{i=1}^{k} Z_i^2} \right)^{\frac{1}{2}}. \tag{4.2}$$

Now substituting (4.1) into (4.2) gives the remarkably simple bound as, for $k \geq 2$,

$$\langle \psi | H | \psi \rangle = T - 2e^2 \sqrt{\pi} \left(N + \sum_{i=1}^{k} Z_i^2 \right)^{\frac{1}{2}} \left(\int d^2\mathbf{x} \rho^2(\mathbf{x}) \right)^{\frac{1}{2}}. \tag{4.3}$$

Equation (4.3) suggests us to use the lower bound for the kinetic energy, T, in some power of an integral of $\rho^2(\mathbf{x})$. Substituting (2.12) into (4.3), the lower bound for the ground state energy of the fermionic matter in two dimensions is expressed as, for $k \geq 2$,

$$\begin{aligned}\langle\psi|H|\psi\rangle &\geq \frac{\pi\hbar^2}{2\overline{\zeta}\,m(1+\delta)}\int d^2\mathbf{x}\,\rho^2(\mathbf{x}) - 2e^2\sqrt{\pi}\left(N+\sum_{i=1}^{k}Z_i^2\right)^{\frac{1}{2}}\left(\int d^2\mathbf{x}\,\rho^2(\mathbf{x})\right)^{\frac{1}{2}} \\ &= \frac{\pi\hbar^2}{2\overline{\zeta}\,m(1+\delta)}\left[\left(\int d^2\mathbf{x}\,\rho^2(\mathbf{x})\right)^{\frac{1}{2}} - e^2\sqrt{\pi}\left(N+\sum_{i=1}^{k}Z_i^2\right)^{\frac{1}{2}}\right]^2 \\ &\quad - e^4\pi\overline{\zeta}\left(\frac{2m(1+\delta)}{\pi\hbar^2}\right)\left(N+\sum_{i=1}^{k}Z_i^2\right).\end{aligned}$$

Taking δ be arbitrary small and $\delta \ll N$ yields

$$\langle\psi|H|\psi\rangle > -4\overline{\zeta}\left(\frac{me^4}{2\hbar^2}\right)N\left(1+\frac{\sum_{i=1}^{k}Z_i^2}{N}\right). \tag{4.4}$$

Now, we note the bound

$$\sum_{j=1}^{k}Z_j^2 \leq Z_{MAX}\sum_{j=1}^{k}Z_j = Z_{MAX}N \tag{4.5}$$

where Z_{MAX} corresponds to the nucleus with largest charge in unit of $|e|$. Substituting (4.5) into (4.4), we finally obtain the lower bound for the ground state energy of fermionic matter in two dimensions as

$$\langle\psi|H|\psi\rangle > -c_F(\overline{\zeta})N$$

where $c_F = 4\overline{\zeta}\left(1+Z_{MAX}\right)\left(\dfrac{me^4}{2\hbar^2}\right).$

Conclusion and Discussion

We obtain an N behavior which is to be compared to the N one in three dimensions of Dyson [2], Lenard [6], Lieb [7,8], Lieb and Thirring [9], and Manoukian and Muthaporn [11]. The ground state energy for fermionic matter will grow not slower than $-N$ and is obviously quite relevant physically to the stability of matter. A power law behavior with N power one implies that stability as the formation of a single system consisting of $\left(2N+2N\right)$ particles is favoured over two separate systems brought together, each consisting of $\left(N+N\right)$ particles, and on energy released upon the collapse of the two systems into one, being proportional to $\left[(2N)-2(N)\right]=0$, will be overwhelming large for realistic large N, e.g., $N\sim 10^{23}$. It leads to the conclusion that as more and more matter is put together, matter will inflate.

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