บทความวิจัย

การยุบตัวของสสารที่ไม่เป็นไปตามหลักการกีดกันใน 2 มิติ

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บทคัดย่อ

ความสัมพันธ์ระหว่างจำนวนของอนุภาคประจุลบ (N) และรัศมี (R) ของสสารที่ไม่เป็นไปตามหลักการกีดกัน ภายใต้ปฏิกิริยาคูลอมบ์ใน 2 มิติ แสดงให้เห็นว่า หากเรานำสสารประเภทนี้มารวมกันเป็นจำนวนมากๆ ใน 2 มิติ สสารจะยุบตัว และเมื่อพิจารณาจากความน่าจะเป็นของการมีอนุภาคประจุลบภายใต้ปฏิกิริยาคูลอมบ์ภายในวงกลม พบว่า R ไม่สามารถหดตัวเร็วกว่า $N^{-1/2}$ เมื่อ N มีค่ามาก

คำสำคัญ: พลังงานสถานะพื้น ความไม่เสถียร สสารประเภทโบซอน การยุบตัว

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Deflation of Matter without Exclusion Principle in Two Dimensions

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ABSTRACT

The relationship between number N of negatively charge particle and radius R of matter without exclusion principle in this paper implies that the matter will deflate if we put more and more such matter together. For a non-vanishing probability of having the negatively charged particle, with Coulomb interactions, within a circle of radius, necessarily cannot shrink faster than $N^{-1/2}$ for large N

Keywords: ground state energy, instability, lower bound, bosonic matter

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Introduction

The key result in the problem of the instability of matter without exclusion principle "bosonic matter" as well as stability of matter with exclusion principle "fermionic matter", is the N^{α}, $\alpha \geq 1$ law behavior $E_N \sim N^{\alpha}$ of the ground-state energy, has been carried out in the early classic work [1-5]. In case of $\alpha = 1$ matter is stable, Manoukian and Sirininlakul [6] have recently shown that for a non-vanishing probability of having electrons in matter, with Coulomb interactions, within a sphere of radius *R*, the latter necessarily grows not any slower than $N^{1/3}$ for large, *N* where *N* denotes the number of electrons. This conclusion is based on in terms of derived inequality relating to the probability for the electrons lying within such a sphere, the volume v_R of the latter and the number *N* of electrons. In other case, such a power law behavior with $\alpha > 1$ impliying that the formation of single system (in three dimensions) consisting of $(2N + 2N)$ particles is favored over two separate systems brought together each consisting of $(N + N)$ particles describing that the energy will be released upon the deflation of the two systems into one. What conclusion can be draw about this matter if the dimensions are changed from three to two? In case of two dimensions, there have been much interested in the ground-state energy of bosonic matter during recent year, e.g. [7,8,9]. As a matter of fact, it is an important theoretical question to investigate if the change of dimensions of space will change matter from unstable to stable phase. The purpose of this paper is to prove rigorously for a non-vanishing probability of having negatively charge particles in matter within a circle of radius *R*. The relationship between *N* and *R* will lead to conclude that bosonic matter is unstable (deflation) or stable (inflation) for large *N*.

Upper Bound for $\int \mathrm{d}^2\vec{x}\rho^2(\vec{x})$

We first derive an upper bound to the expectation value of the kinetic energy of the negatively charged particles. We define the particle density of *N* (spin 0) bosons:

$$\rho(\vec{x}) = \int \mathrm{d}^2\vec{x}_2 \ldots \mathrm{d}^2\vec{x}_N \left|\Psi(\vec{x},\vec{x}_2,\ldots,\vec{x}_N)\right|^2 \tag{1}$$

and $\int \mathrm{d}^2\vec{x}\rho^2(\vec{x}) = N$, Ψ denotes a normalized state giving a strictly negative expectation value of the Hamiltonian,

$$H = \sum_{i=1}^{N}\frac{\vec{p}_i^{\,2}}{2m} + \sum_{i<j}^{N}\frac{e^2}{\left|\vec{x}_i - \vec{x}_j\right|} + \sum_{i<j}^{k}\frac{Z_i Z_j e^2}{\left|\vec{R}_i - \vec{R}_j\right|} - \sum_{i=1}^{N}\sum_{j=1}^{k}\frac{Z_j e^2}{\left|\vec{x}_i - \vec{R}_j\right|} \tag{2}$$

where m denote the mass of negatively charged particle, denote atomic number and $\vec{x}$, $\vec{R}$ correspond, respectively, to positions of negatively charged particles and nuclei in two dimensions. We have also considered neutral matter $\sum_{i=1}^{k} Z_i = N$.

Let $|\Psi(m)\rangle$ denote a normalized state giving a strictly negative expectation value for the Hamiltonian, i.e.,

$$-\varepsilon_N[m] \leq \langle \Psi(m) | H | \Psi(m) \rangle < 0 \tag{3}$$

where $-\varepsilon_N[m] = E_N < 0$ is the ground-state energy. The negative spectrum of H easily follows by noting that $-\varepsilon_N[m]$ is bounded above by $-(me^4/2\hbar^2)\sum_{i=1}^{k} Z_i^2$, and we have emphasized its dependence on the mass of the negatively charged particle. To establish the statement made above, we need upper and lower bounds to the expectation value of the kinetic energy operator

$$T = \left\langle \Psi(m) \left| \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} \right| \Psi(m) \right\rangle. \tag{4}$$

Here we note, in general, that a part of a negative spectrum does not necessarily coincide with bound states. By definition of the ground-state energy, the state $|\Psi(m/2)$ cannot lead for $\langle\Psi(m/2)|H|\Psi(m/2)$ a numerical value lower than $-\varepsilon_N[m]$. That is

$$-\varepsilon_N[m] \leq \langle \Psi(m/2) | H | \Psi(m/2) \rangle \tag{5}$$

we note that the interaction part V of the Hamiltonian H in (2) is not explicitly dependent on m:

$$V = \sum_{i<j}^{N} \frac{e^2}{|\vec{x}_i - \vec{x}_j|} + \sum_{i<j}^{k} \frac{Z_i Z_j e^2}{|\vec{R}_i - \vec{R}_j|} - \sum_{i=1}^{N} \sum_{j=1}^{k} \frac{Z_j e^2}{|\vec{x}_i - \vec{R}_j|}. \tag{6}$$

Substituting (6) into the right-hand side of (2), we can rewrite (3) as

$$-\varepsilon_N[m] \leq \left\langle \Psi(m/2) \left| \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + V \right| \Psi(m/2) \right\rangle. \tag{7}$$

By using (7), replace m by $2m$ we obtain

$$-\varepsilon_N[2m] \leq \left\langle \Psi(m) \left| \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{4m} + V \right| \Psi(m) \right\rangle. \tag{8}$$

From (2) and (6), we may also rewrite the Hamiltonian as

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{4m} + \left(\sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{4m} + V \right). \tag{9}$$

The extreme right-hand side of the inequality (3) then leads to

$$\langle\Psi(m)|\sum_{i=1}^{N}\frac{\vec{p}_i^{\,2}}{4m}|\Psi(m)\rangle < -\langle\Psi(m)|\sum_{i=1}^{N}\frac{\vec{p}_i^{\,2}}{4m}+V|\Psi(m)\rangle \le \varepsilon_N[2m]. \tag{10}$$

Multiply both sides of (10) by 2, to obtain

$$T < 2\varepsilon_N[2m]. \tag{11}$$

From [9], we obtain the lower for the ground-state energy of N bosons:

$$\langle\Psi(m)|H|\Psi(m)\rangle = -\varepsilon_N[m] > -4\left(\frac{me^4}{2\hbar^2}\right)N^2\left(1+\sum_{i=1}^{k}Z_{\max}\right). \tag{12}$$

where Z_{max} corresponds to the nucleus of having the charge.

Replace m by $2m$ to obtain

$$\varepsilon_N[2m] < 4\left(\frac{me^4}{\hbar^2}\right)N^2\left(1+\sum_{i=1}^{k}Z_{\max}\right). \tag{13}$$

An adaptation of the Schwinger bound [10] for matter without exclusion in two dimensions, then leads to

$$T \ge \frac{\pi}{N}\frac{\hbar^2}{2m}\int d^2\vec{x}\,\rho^2(\vec{x}), \tag{14}$$

by using, in the process, that due to the bose character of the N particles, they may all be put in the lowest energy level which accounts for the N^{-1} factor on the right-hand side of the inequality.

From (11), (13) and (14), we obtain

$$\int d^2\vec{x}\,\rho^2(\vec{x}) < 16N^3\left(\frac{m^2e^4}{\pi\hbar^4}\right)\left(1+\sum_{i=1}^{k}Z_{\max}\right) \tag{15}$$

where, from the normalization condition

$$\int d^2\vec{x}\rho^2(\vec{x}) = N \tag{16}$$

Deflation of matter without exclusion principle

To investigate the deflation of matter, we are interested in theexpression which gives the probability of finding all the negatively charged particles within the circle of radius R let $\vec{x}$ denote the position (in two dimensions) of a negatively charged particle relative, for example, to the center of mass of the nuclei. Then clearly for the probability of the negatively charged particles to lie within such a circle, let $\chi_R(\vec{x}) = 1$, if $\vec{x}$ if lines within a circle of radius R and = 0 otherwise. We have

$$\mathrm{Prob}\left[\left|\vec{x}_1\right| \le R,\ldots,\left|\vec{x}_N\right| \le R\right] \le \mathrm{Prob}\left[\left|\vec{x}_1\right| \le R\right] \;= \frac{1}{N}\int \mathrm{d}^2\vec{x}\,\chi_R\left(\vec{x}\right)\rho\left(\vec{x}\right) \tag{17}$$

where in the last inequality, we have used Holder's inequality, to obtain

$$\int \mathrm{d}^2\vec{x}\,\chi_R\left(\vec{x}\right)\rho\left(\vec{x}\right) \le \left(\int d^2\vec{x}\rho^2\left(\vec{x}\right)\right)^{1/2} A_R^{1/2} \tag{18}$$

where $\chi_R^2(\vec{x}) = \chi_R(\vec{x})$, and

$$\int \mathrm{d}^2\vec{x}\,\chi_R\left(\vec{x}\right) = A_R = \pi R^2 \tag{19}$$

with A_R denoting the area of a circle of radius

From (17), we obtain

$$\mathrm{Prob}\left[\left|\vec{x}_1\right| \le R,\ldots,\left|\vec{x}_N\right| \le R\right] \le \frac{A_R^{1/2}}{N}\left(\int \mathrm{d}^2\vec{x}\,\rho^2\left(\vec{x}\right)\right)^{1/2} \tag{20}$$

and from (15), we obtain the following bound

$$\left(\int \mathrm{d}^2\vec{x}\,\rho^2\left(\vec{x}\right)\right)^{1/2} < \frac{2.2563N^{3/2}}{a_0}\left(1+Z_{\max}\right)^{1/2} \tag{21}$$

where $a_0 = \mathrm{h}^2/me^2$ is the Bohr radius and $\sqrt{16/\pi} \approx 2.2563$.

Substitute (21) into the right-hand side of the equalities (20), to obtain

$$\mathrm{Prob}\left[\left|\vec{x}_1\right| \le R,\ldots,\left|\vec{x}_N\right| \le R\right] < \frac{2.2563\left(A_R N\right)^{1/2}}{a_0}\left(1+Z_{\max}\right)^{1/2}. \tag{22}$$

From (22), we then have the main result of this paper :

$$\mathrm{Prob}\left[\left|\vec{x}_1\right| \le R,\ldots,\left|\vec{x}_N\right| \le R\right]\left(\frac{1}{A_R N}\right)^{1/2} < \frac{2.2563}{a_0}\left(1+Z_{\max}\right)^{1/2}. \tag{23}$$

This will be used in the next section to obtain a lower bound to measure of the extension of matter.

Non-zero lower Bound for a Measure of the Extension of Matter

As a measure of the extension of matter, we introduce the expectation value :

$$\left\langle \sum_{i=1}^{N}\frac{\left|\vec{x}_i\right|}{N}\right\rangle = \int \mathrm{d}^2\vec{x}_1\ldots\mathrm{d}^2\vec{x}_N\left(\sum_{i=1}^{N}\frac{\left|\vec{x}_i\right|}{N}\right)\left|\Psi(\vec{x}_1,\ldots,\vec{x}_N)\right|^2 = \frac{1}{N}\int \mathrm{d}^2\vec{x}\left|\vec{x}\right|\rho(\vec{x}). \tag{24}$$

Now we use the fact that

$$\frac{1}{N}\int d^2\vec{x}\,|\vec{x}|\,\rho(\vec{x}) > \frac{R}{N}\int_{|\vec{x}|>R} d^2\vec{x}\,\rho(\vec{x})$$

$$= R\,\mathrm{Prob}[|\vec{x}| > R] \tag{25}$$

and

$$\mathrm{Prob}[|\vec{x}| > R] == 1 - \mathrm{Prob}[|\vec{x}| \le R] \tag{26}$$

From (19), (24) and (26) we then the bounds

$$\left\langle \sum_{i=1}^{N} \frac{|\vec{x}_i|}{N} \right\rangle > R\left[1 - \frac{2.2563\left(A_R N\right)^{1/2}}{a_0}\left(1+Z_{\max}\right)^{1/2}\right]. \tag{27}$$

Upon optimizing the left-hand side of the above inequality over R, we get

$$0 > 1 - \frac{4.5126R\left(\pi N\right)^{1/2}}{a_0}\left(1+Z_{\max}\right)^{1/2} \tag{28}$$

Leading to

$$R > \frac{1}{4.5126}\left(\frac{1}{\pi N}\right)^{1/2}\frac{a_0}{(1+Z_{\max})^{1/2}} \tag{29}$$

Substitute (29) into the right-hand side of inequality (26), we obtain the explicit non-zero lower bound

$$\left\langle \sum_{i=1}^{N} \frac{|\vec{x}_i|}{N} \right\rangle > \frac{1}{4.5126}\left(\frac{1}{\pi N}\right)^{1/2}\frac{a_0}{(1+Z_{\max})^{1/2}}. \tag{30}$$

Conclusion and Discussion

From the main result of this paper, (23), it leads to the unescapable conclusion that necessity for a non-vanishing probability of having the negatively charged particles within an area A_R of circle, the corresponding radius R, cannot shrink faster than $N^{-1/2}$ for large N ($N \to \infty$), since otherwise the left-hand side of (23) would go to infinity and would be in contradiction with the finite upper bound on its right-hand side, thus establishing the above stated result. This relationship between N and R above show that matter without the exclusion principle in two dimensions will deflate into a condensed high-density phase when we put more and more this matter together, and lead to say that "system is unstable". This is regarding to a deflating Dyson states [1]: "The assembly of any two macroscopic objects would release energy comparable to that of an atomic bomb..."

Methods similar to the ones developed above can use to study the localizability and instability of other quantum mechanical systems. We note that the inequality in (23) is sufficient to reach such a conclusion but cannot establish the actual deflation of such matter. This formidable problem will be attempted in a future report.

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