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Research Article

A Comparative Study of Confidence Intervals for the Parameter of the Inverse Xgamma Distribution

Nawalax Thongjub[1] and Wararit Panichkitkosolkul[1,2*]

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ABSTRACT

This study develops and evaluates four confidence interval (CI) procedures for the parameter of the inverse Xgamma (IXG) distribution: the likelihood-based, Wald-type, bootstrap-t, and bias-corrected and accelerated (BCa) bootstrap CIs. A Monte Carlo simulation study was conducted over a range of parameter values and sample sizes to assess their finite-sample performance in terms of empirical coverage probability and average width. The results show that the likelihood-based and Wald-type intervals generally achieved empirical coverage probabilities close to the nominal 0.95 level across most settings. The bootstrap-t and BCa bootstrap intervals tended to produce shorter intervals in some small-sample settings but exhibited noticeable undercoverage. Their coverage performance improved as the sample size increased, and the differences among the four methods became less pronounced for large samples. The proposed methods were also applied to failure-time data from 15 electronic components subjected to an accelerated life test. The IXG distribution provided the smallest Akaike and Bayesian information criterion values among the competing lifetime models considered. In addition, the Kolmogorov-Smirnov test and the probability-probability plot supported the adequacy of the fitted IXG distribution. For the real dataset, the likelihood-based and Wald-type intervals were narrower and relatively similar, whereas the bootstrap-t and BCa bootstrap intervals were wider and more asymmetric, reflecting greater small-sample uncertainty. Overall, the likelihood-based and Wald-type procedures provided the most favorable balance between coverage accuracy and interval width, particularly for small and moderate sample sizes.

Keywords: Likelihood-ratio interval, Wald approximation, Studentized bootstrap, Monte Carlo simulation, Lifetime modeling

[1] Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Pathum Thani 12120, Thailand

[2] Thammasat University Research Unit in Mathematical Sciences and Applications, Pathum Thani 12120, Thailand

*Corresponding author, email: wararit@mathstat.sci.tu.ac.th

Introduction

Lifetime data in reliability engineering, survival analysis, environmental science, and medical research frequently exhibit positive values, a right-skewed distribution, and hazard rates that vary significantly over time. Although classical lifetime distributions such as the exponential and gamma distributions are widely used for modeling nonnegative and skewed data, their hazard-rate structures may be too restrictive for some applications. In particular, the exponential distribution assumes a constant hazard rate, whereas real failure-time data may exhibit more complex, non-monotonic patterns. Consequently, more flexible lifetime distributions are often needed when the underlying failure mechanism cannot be adequately represented by simple monotonic hazard structures.

The inverse Xgamma (IXG) distribution, introduced by Yadav et al. [1], provides an alternative one-parameter lifetime distribution for positively skewed data. A significant characteristic of the IXG distribution is its capacity to support a non-monotonic, upside-down bathtub-shaped hazard rate, which may be useful in applications where the likelihood of failure first increases and then decreases. This flexibility makes the IXG distribution a potentially useful model in reliability and lifetime-data analysis. The IXG distribution is obtained via an inverse transformation of the Xgamma distribution, offering more adaptability relative to its baseline model, therefore making it appropriate for the analysis of real-world data in reliability and medical research.

Despite its attractive properties, statistical inference concerning the parameter of the IXG distribution is still somewhat limited. Previous studies have mainly concentrated on point-estimating methods such as maximum likelihood (ML) estimation, and least squares estimation, in conjunction with asymptotic confidence intervals (CIs) derived from specific estimators. However, comprehensive investigation and comparison of multiple CI construction methods for the IXG parameter remain insufficient.

CIs are essential tools used in statistical inference because they offer a possible range of values for unknown parameters, rather than relying on a single point estimate [2, 3]. Despite the fact that point estimators are beneficial for summarizing sample information, they do not accurately represent the uncertainty that results from sampling variation. This limitation is overcome by CIs, which incorporate both the approximated parameter value and its associated variability, thereby providing a more informative foundation for statistical interpretation. A well-constructed CI enables researchers to evaluate the reliability of inferential conclusions, assess the precision of an estimator, and make decisions with a nominal confidence level in practical applications.

Accurate interval estimation is of significance in the analysis of lifetime data, where statistical models are frequently applied to summarize reliability characteristics, survival times, failure systems, and waiting times. The uncertainty in parameter estimation can significantly influence statistical inference and practical decision-making, as model parameters frequently determine important distributional properties, including the shape of the density function, tail behavior, and hazard rate pattern. As a result, CIs are essential instruments for quantifying the uncertainty of estimation and determining the precision of point estimates.

Several methods have been established for the construction of CIs, each based on different theoretical principles and assumptions. Classical methods, such as the likelihood-based CI and the Wald-type CI, are derived from asymptotic properties of ML estimator and are widely used because of their computational simplicity [4, 5]. However, their performance may be inadequate in small samples or when the underlying sampling distribution deviations from normality. Resampling-based techniques, such as the bias-corrected and

accelerated (BCa) bootstrap CI and the bootstrap-t CI, have been proposed to address this limitation. These methods typically offer increased flexibility and may enhance finite-sample performance by reducing the dependence on asymptotic approximations.

Although CI estimation has been extensively investigated for many lifetime distributions, relatively little attention has been given to the IXG distribution. To our knowledge, current research has focused solely on asymptotic confidence intervals linked to particular estimators, but other interval estimation methods have not been comprehensively compared. Since different CI methods may exhibit substantially different performance in terms of empirical coverage probability and interval average width, particularly for skewed distributions and small-moderate sample sizes, a comprehensive investigation is necessary to identify reliable inferential procedures for the IXG distribution.

Motivated by this gap in the literature, the present study aims to develop and compare four CI methods for the parameter of the IXG distribution, namely the likelihood-based CI, Wald-type CI, bootstrap-t CI, and BCa bootstrap CI. Their performance is evaluated using comprehensive Monte Carlo simulation experiments using empirical coverage probability and average width as evaluation criteria. In addition, the practical applicability of the proposed methods is illustrated through the analysis of real datasets. The findings of this study aim to offer valuable guidance for practitioners and researchers in choosing suitable interval estimate methods for the IXG distribution and to enhance the existing literature on statistical inference for lifetime distributions.

Methods

This section provides an overview of the IXG distribution, with emphasis on point estimation of its parameter and the construction of CIs. The details include the maximum likelihood estimation for obtaining the point estimate, as well as four methods for constructing CIs to assess the precision and reliability of the estimated parameter.

The Inverse Xgamma Distribution and Its Properties

Yadav et al. [1] introduced the inverse Xgamma (IXG) distribution as a one-parameter lifetime model obtained by applying the reciprocal transformation to the Xgamma distribution proposed by Sen et al. [6]. Specifically, let Y follow an Xgamma distribution with parameter $\theta > 0,$ and define $X = 1/Y.$ Then, X is said to follow an inverse Xgamma distribution, denoted by $X \sim \mathrm{IXG}(\theta).$ The probability density function of X is

$$f(x;\theta) = \frac{\theta^2}{(1+\theta)}\frac{1}{x^2}\left(1+\frac{\theta}{2x^2}\right)\exp\left(-\frac{\theta}{x}\right), \quad x>0,\ \theta>0$$

and its cumulative distribution function is

$$F(x;\theta) = \left(1+\frac{\theta}{(1+\theta)x}+\frac{\theta^2}{2(1+\theta)x^2}\right)\exp\left(-\frac{\theta}{x}\right), \quad x>0,\ \theta>0$$

The IXG distribution is unimodal and positively skewed, making it suitable for modelling nonnegative observations with substantial right skewness. An important characteristic of the distribution is its ability to accommodate a non-monotonic, upside-down bathtub-shaped hazard rate. This feature provides greater flexibility

than distributions restricted to constant or monotonic hazard rates and makes the IXG distribution potentially useful in reliability engineering, survival analysis, clinical studies, and accelerated life-testing applications.

The survival function of the IXG distribution is given by

$$S(x;\theta) = 1 - F(x;\theta) = 1 - \left(1 + \frac{\theta}{(1+\theta)x} + \frac{\theta^2}{2(1+\theta)x^2}\right)\exp\left(-\frac{\theta}{x}\right)$$

whereas the hazard rate function can be expressed as

$$h(x;\theta) = \frac{f(x;\theta)}{S(x;\theta)} = \frac{\frac{\theta^2}{(1+\theta)}\frac{1}{x^2}\left(1 + \frac{\theta}{2x^2}\right)\exp\left(-\frac{\theta}{x}\right)}{1 - \left(1 + \frac{\theta}{(1+\theta)x} + \frac{\theta^2}{2(1+\theta)x^2}\right)\exp\left(-\frac{\theta}{x}\right)}$$

The graphical results reported in the original study show that the survival probability decreases with time, while the hazard function initially increases and subsequently decreases. Therefore, the IXG distribution is particularly relevant when the risk of failure or the occurrence of an event rises during an early period and then declines.

An important mathematical property of the IXG distribution is that its ordinary positive moments do not exist. Consequently, conventional moment-based summaries, such as the arithmetic mean, variance, moment-based skewness, and kurtosis, are not available. The distributional characteristics can be analyzed using inverse moments and quantile-based measures. The r^{th} inverse moment is

$$\mu_r'^{(-1)} = E\left(X^{-r}\right) = \frac{\theta^2}{2(1+\theta)}\left[\frac{2\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+3)}{\theta^{r+2}}\right], \quad r = 1, 2, 3, \ldots$$

In particular,

$$E\left(\frac{1}{X}\right) = \frac{\theta+3}{\theta(1+\theta)}$$

Because the quantile function has no explicit closed-form expression, the p^{th} quantile $Q(p)$ must be obtained numerically by solving

$$p = \left(1 + \frac{\theta}{(1+\theta)Q(p)} + \frac{\theta^2}{2(1+\theta)Q^2(p)}\right)\exp\left(-\frac{\theta}{Q(p)}\right), \quad 0 < p < 1$$

Quantile-based measures, such as Bowley's skewness [7] and Moors' kurtosis [8], are therefore more appropriate than classical moment-based measures for describing the shape of the IXG distribution. The numerical results in the original article indicate that the distribution is highly positively skewed.

Point Parameter Estimation

Let $X = (X_1, ..., X_n)$ be independent and identically distributed random variables from the IXG distribution with parameter θ, where n denotes the sample size and $\mathbf{x} = (x_1, ..., x_n)$ represent the corresponding observed sample. The point estimator for the parameter of the IXG distribution can be derived using the ML method. This method involves constructing the likelihood function based on a given sample and then determining the parameter value that maximizes this function, as outlined in the following steps.

Step 1: Find the likelihood function

The likelihood function $L(\theta;\mathbf{x})$ represents the joint probability of observing a random sample drawn from the IXG distribution. It is defined as:

$$L(\theta;\mathbf{x})=\prod_{i=1}^{n} f(x_i;\theta)=\prod_{i=1}^{n}\left[\frac{\theta^2}{(1+\theta)}\frac{1}{x_i^2}\left(1+\frac{\theta}{2x_i^2}\right)\exp\left(-\frac{\theta}{x_i}\right)\right]$$

Step 2: Find the log-likelihood function

Due to the mathematical complication involved in directly differentiating the likelihood function, the log-likelihood function is employed to facilitate the differentiation process. The log-likelihood function is given by

$$\ell(\theta;\mathbf{x})=\log L(\theta;\mathbf{x})=2n\log\theta-n\log(1+\theta)-2\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\log\left(1+\frac{\theta}{2x_i^2}\right)-\theta\sum_{i=1}^{n}\frac{1}{x_i}$$

Step 3: Differentiate the log-likelihood function

To determine the value of θ that maximizes the log-likelihood function, the derivative of $\log L(\theta;\mathbf{x})$ with respect to θ is computed. This yields the score function $U(\theta;\mathbf{x})$, which is expressed as follows:

$$U(\theta;\mathbf{x})=\frac{\partial}{\partial\theta}\ell(\theta;\mathbf{x})=\sum_{i=1}^{n}\left(\frac{1}{2x_i^2+\theta}\right)+\frac{2n}{\theta}-\frac{n}{1+\theta}-\sum_{i=1}^{n}\frac{1}{x_i}$$

Step 4: Set the derivative equal to zero and solve for the ML estimator

The value of θ is obtained by solving the equation $U(\theta;\mathbf{x})\overset{\text{set}}{=}0$, which involves setting the score function to zero. This method determines the critical points that may correspond to potential maxima of the log-likelihood function. Given the absence of a closed-form solution for the ML estimator of the parameter, numerical iterative methods are employed to address the resulting non-linear optimization problem [9]. In this study, ML estimation was conducted using the Newton-Raphson method, as implemented using the maxLik package [10] in R version 4.6.0 [11].

Confidence Intervals

This subsection presents four methods for constructing CIs for the parameter of the IXG distribution: the likelihood-based CI, the Wald-type CI, the bootstrap-t CI, and the BCa bootstrap CI.

Likelihood-based Confidence Interval

The likelihood function for the IXG distribution is a function of the parameter θ, given the observed data **x.** It represents the probability of observing the given data under various possible values of θ**.** After solving $U(\theta;\mathbf{x})\overset{\text{set}}{=}0$, the ML estimator of θ will be obtained, and this is the most plausible estimate given the observed data. This ML estimator is then used to construct the likelihood-based CI. The process begins by defining a likelihood ratio $\lambda(\theta)$ as $\lambda(\theta)=L(\theta;\mathbf{x})\big/L(\hat{\theta};\mathbf{x})$**.** Based on the Wilks' theorem [4], $-2\log\lambda(\theta)$ follows approximately a chi-square distribution with degrees of freedom equal to the number of parameters being estimated. Therefore, the CI for θ at $(1-\alpha)100\%$ confidence level is given by

$$\left\{\theta\left|-2\log\frac{L(\theta;\mathbf{x})}{L(\hat{\theta};\mathbf{x})}\le\chi^2_{1-\alpha,1}\right.\right\}=$$

$$\left\{\theta\left|-2\log\left[\frac{\theta^{2n}\left(1+\hat{\theta}\right)^n}{\hat{\theta}^{2n}\left(1+\theta\right)^n}\cdot\frac{\prod_{i=1}^{n}\frac{1}{x_i^2}\left(1+\frac{\theta}{2x_i^2}\right)}{\prod_{i=1}^{n}\frac{1}{x_i^2}\left(1+\frac{\hat{\theta}}{2x_i^2}\right)}\exp\left(-\theta\sum_{i=1}^{n}\frac{1}{x_i}+\hat{\theta}\sum_{i=1}^{n}\frac{1}{x_i}\right)\right]\le\chi^2_{1-\alpha,1}\right.\right\}$$

where $\chi^2_{1-\alpha,1}$ is the $(1-\alpha)$-quantile of the chi-square distribution with 1 degree of freedom. Further details on likelihood-based inference and Wilks' theorem can be found in Pawitan [4] and Severini [5].

Figure 1 presents the plot of $-2\log\lambda(\theta)$ against θ, together with the quantile $\chi^2_{0.95,1}$ and the corresponding 95% likelihood-based CI, based on a random sample of size 20 from the IXG distribution with θ=1.

The cut-off point used to construct a likelihood-based CI is often determined from an asymptotic distribution, such as the chi-square distribution. Therefore, the accuracy of the likelihood-based CI in covering the true parameter value depends on whether the sample size is sufficiently large for the asymptotic approximation to be valid. However, likelihood-based CIs are not necessarily restricted to large-sample settings. They may still provide accurate interval estimates for smaller samples, provided that the likelihood function is sufficiently regular and well behaved.

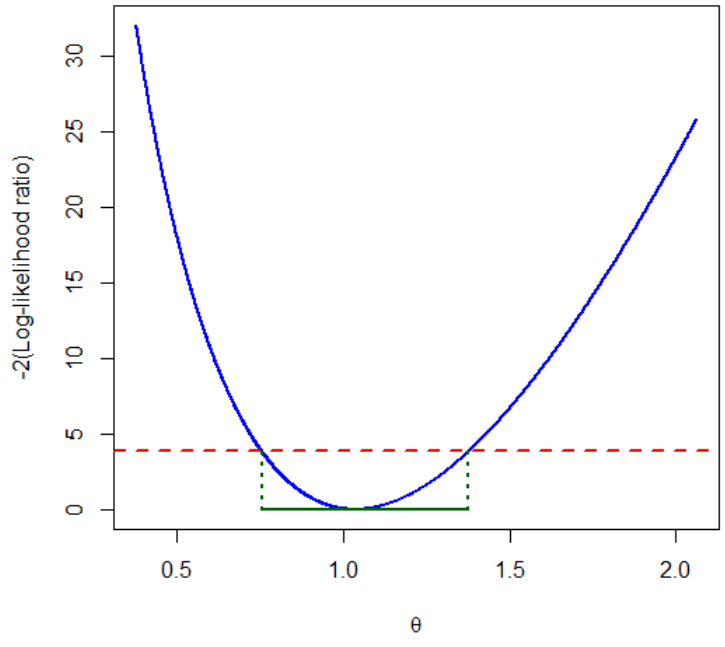


Figure 1 The plot of $-2\log\lambda(\theta)$ versus θ.

Wald-type Confidence Interval

Under the standard regularity conditions for ML estimation, the distribution of the ML estimator can be approximated asymptotically by

$$\hat{\theta}\overset{a}{\sim}N\left(\theta,I_n^{-1}(\theta)\right)$$

where $I_n(\theta)$ denotes the Fisher information based on a sample of size n. In practice, the variance of $\hat{\theta}$ can be estimated using the inverse of the observed information evaluated at $\hat{\theta}$ [2, 4-5]. The second derivative of the log-likelihood function is

$$\frac{\partial^2 \ell(\theta;\mathbf{x})}{\partial \theta^2} = -\frac{2n}{\theta^2} + \frac{n}{(1+\theta)^2} - \sum_{i=1}^{n} \frac{1}{(2x_i^2+\theta)^2}$$

Therefore, the observed information function is

$$J_n(\theta) = -\frac{\partial^2 \ell(\theta;\mathbf{x})}{\partial \theta^2} = \frac{2n}{\theta^2} - \frac{n}{(1+\theta)^2} + \sum_{i=1}^{n} \frac{1}{(2x_i^2+\theta)^2}$$

Accordingly, the estimated variance and standard error of $\hat{\theta}$ are given by

$$\mathrm{Var}(\hat{\theta}) = \left[J_n(\hat{\theta}) \right]^{-1} \quad \text{and} \quad \mathrm{SE}(\hat{\theta}) = \sqrt{\mathrm{Var}(\hat{\theta})}$$

respectively. Therefore, an approximate $(1-\alpha)100\%$ Wald-type CI for θ is

$$\left(\hat{\theta} - z_{\alpha/2}\mathrm{SE}(\hat{\theta}),\ \hat{\theta} + z_{\alpha/2}\mathrm{SE}(\hat{\theta}) \right)$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution. Equivalently, the interval can be written explicitly as

$$\mathrm{CI_W}(\theta) = \left(\hat{\theta} - z_{\alpha/2} \sqrt{\left[\frac{2n}{\hat{\theta}^2} - \frac{n}{(1+\hat{\theta})^2} + \sum_{i=1}^{n} \frac{1}{(2x_i^2+\hat{\theta})^2} \right]^{-1}},\ \hat{\theta} + z_{\alpha/2} \sqrt{\left[\frac{2n}{\hat{\theta}^2} - \frac{n}{(1+\hat{\theta})^2} + \sum_{i=1}^{n} \frac{1}{(2x_i^2+\hat{\theta})^2} \right]^{-1}} \right)$$

The conventional Wald interval may occasionally result in a negative lower endpoint since the parameter space is restricted to $\theta > 0$, especially for small samples or when the estimated standard error is rather large. In the present simulation study, no truncation or transformation was applied to the lower endpoint. Negative lower endpoints, when they occurred, were retained as generated by the Wald-type method so that its empirical performance could be evaluated without post-processing.

Bootstrap-t Confidence Interval

The bootstrap-t CI is a studentized bootstrap procedure that incorporates an estimate of the standard error into the bootstrap statistic. By taking into account differences in the estimated standard error between bootstrap samples, this studentization can increase the interval's accuracy [12, 13].

Let $X_1, X_2, ..., X_n$ be a random sample from the IXG distribution with parameter $\theta > 0$, and let $\hat{\theta}$ denote the ML estimate obtained from the observed sample. The estimated standard error of $\hat{\theta}$ based on the observed information function, is given by

$$\mathrm{SE}(\hat{\theta}) = \sqrt{\left[\frac{2n}{\hat{\theta}^2} - \frac{n}{(1+\hat{\theta})^2} + \sum_{i=1}^{n} \frac{1}{(2x_i^2+\hat{\theta})^2} \right]^{-1}}$$

To construct the bootstrap-t CI, generate B parametric bootstrap samples,

$$X_1^{*(b)}, X_2^{*(b)}, ..., X_n^{*(b)} \sim \mathrm{IXG}(\hat{\theta}), \quad b = 1, 2, ..., B.$$

For the b^{th} bootstrap sample, obtain the bootstrap ML estimate $\hat{\theta}^{*(b)}$ by numerically maximizing the corresponding bootstrap log-likelihood function. The estimated standard error of $\hat{\theta}^{*(b)}$ is

$$\mathrm{SE}\left(\hat{\theta}^{*(b)}\right)=\left[\frac{2n}{\left(\hat{\theta}^{*(b)}\right)^2}-\frac{n}{\left(1+\hat{\theta}^{*(b)}\right)^2}+\sum_{i=1}^{n}\frac{1}{\left[2\left(x_i^{*(b)}\right)^2+\hat{\theta}^{*(b)}\right]^2}\right]^{-1/2}$$

The studentized bootstrap statistic for the b^{th} bootstrap replication is defined as

$$T^{*(b)}=\frac{\hat{\theta}^{*(b)}-\hat{\theta}}{\mathrm{SE}\left(\hat{\theta}^{*(b)}\right)},\quad b=1,2,\ldots,B$$

Let $t^{*}_{\alpha/2}$ and $t^{*}_{1-(\alpha/2)}$ denote the empirical $\alpha/2$ and $1-(\alpha/2)$ quantiles, respectively, of the bootstrap statistics

$$T^{*(1)},T^{*(2)},\ldots,T^{*(B)}$$

The approximate $(1-\alpha)100\%$ bootstrap-t CI for θ is then given by

$$\mathrm{CI}_{\text{Boot-t}}(\theta)=\left(\hat{\theta}-t^{*}_{1-(\alpha/2)}\mathrm{SE}(\hat{\theta}),\ \hat{\theta}-t^{*}_{\alpha/2}\mathrm{SE}(\hat{\theta})\right)$$

The ordering of the bootstrap quantiles is reversed in the interval endpoints because the bootstrap-t statistic is inverted to obtain a confidence set for θ. Because the parameter of the IXG distribution is restricted to the positive domain, the bootstrap-t interval may occasionally produce a negative lower endpoint, especially for small samples. In such cases, the lower endpoint may be truncated at zero. Alternatively, the bootstrap-t procedure may be applied to a transformed parameter, such as $\eta=\log\theta,$ and the resulting confidence limits can subsequently be transformed back to the original scale. Similarly, no truncation or transformation was applied to negative lower endpoints of the bootstrap-t CI in the simulation study. All negative lower endpoints were preserved as produced by the bootstrap-t method.

Bias-Corrected and Accelerated Bootstrap Confidence Interval

The bias-corrected and accelerated (BCa) bootstrap CI is an extension of the percentile bootstrap interval that adjusts for both median bias in the bootstrap distribution and skewness in the sampling distribution of the estimator [12, 13]. These corrections are represented by the bias-correction constant and the acceleration constant, respectively. Thus, the BCa interval is transformation-respecting and can enhance finite-sample accuracy when the estimator's distribution is asymmetric.

Let $X_1,X_2,\ldots,X_n$ be a random sample from the IXG distribution with parameter $\theta>0,$ and let $\hat{\theta}$ denote the ML estimate obtained from the observed sample. To construct the BCa bootstrap CI, generate B parametric bootstrap samples,

$$X_1^{*(b)},X_2^{*(b)},\ldots,X_n^{*(b)}\sim\mathrm{IXG}(\hat{\theta}),\quad b=1,2,\ldots,B$$

and compute the corresponding bootstrap ML estimates

$$\hat{\theta}^{*(1)},\hat{\theta}^{*(2)},\ldots,\hat{\theta}^{*(B)}$$

The bias-correction constant, denoted by z_0 is estimated as the proportion of the bootstrap estimates less than the original parameter estimate $\hat{\theta},$

$$z_0=\Phi^{-1}\left[\frac{1}{B}\sum_{b=1}^{B}I\left(\hat{\theta}^{*(b)}<\hat{\theta}\right)\right]$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of the standard normal distribution and $I(\cdot)$ denotes the indicator function. A positive value of z_0 indicates that a relatively large proportion of the bootstrap estimates are smaller than the original estimate, whereas a negative value indicates the opposite.

The acceleration constant is estimated using the jackknife method. Let $\hat{\theta}_{(-i)}$ denote the ML estimate computed after deleting the i^{th} observation from the original sample, for $i = 1, 2, .., n,$ and define the average jackknife estimate as

$$\hat{\theta}_{(\cdot)} = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{(-i)}$$

The acceleration constant is then given by

$$a = \frac{\sum_{i=1}^{n}\left(\hat{\theta}_{(\cdot)} - \hat{\theta}_{(-i)}\right)^3}{6\left[\sum_{i=1}^{n}\left(\hat{\theta}_{(\cdot)} - \hat{\theta}_{(-i)}\right)^2\right]^{3/2}}$$

For a nominal $(1-\alpha)100\%$ confidence level, let $z_{\alpha/2}$ and $z_{1-(\alpha/2)}$ denote the corresponding standard normal quantiles. The adjusted lower and upper cumulative probabilities are defined as

$$\alpha_{\text{L}}^{\text{BCa}} = \Phi\left[z_0 + \frac{z_0 + z_{\alpha/2}}{1 - a\left(z_0 + z_{\alpha/2}\right)}\right]$$

and

$$\alpha_{U}^{\text{BCa}} = \Phi\left[z_0 + \frac{z_0 + z_{1-(\alpha/2)}}{1 - a\left(z_0 + z_{1-(\alpha/2)}\right)}\right]$$

respectively. If $\hat{\theta}^{*}_{(p)}$ denotes the empirical p^{th} quantile of the bootstrap estimates $\hat{\theta}^{*(1)}, \hat{\theta}^{*(2)}, ..., \hat{\theta}^{*(B)}$, then the approximate $(1-\alpha)100\%$ BCa bootstrap CI for θ is

$$\text{CI}_{\text{BCa}}(\theta) = \left(\hat{\theta}^{*}_{\left(\alpha_{\text{L}}^{\text{BCa}}\right)}, \hat{\theta}^{*}_{\left(\alpha_{\text{U}}^{\text{BCa}}\right)}\right)$$

The BCa bootstrap procedure modifies the nominal percentile levels according to both the bias and the rate of change of the standard error of the estimator. When $z_0 = 0$ and $a = 0,$ the adjusted probabilities reduce to $\alpha/2$ and $1-(\alpha/2)$, and the BCa bootstrap interval coincides with the ordinary percentile bootstrap CI. Because the interval endpoints are obtained directly from the empirical distribution of positive bootstrap estimates, the resulting BCa bootstrap interval generally respects the parameter restriction $\theta > 0,$ provided that the numerical estimation procedure is constrained to the admissible parameter space.

Results and Applications

Simulation Results

A Monte Carlo simulation study based on *M*=2000 independent replications was conducted to evaluate and compare the finite-sample performance of four CI procedures for the parameter θ of the IXG distribution. The methods considered were the likelihood-based CI, the Wald-type CI, the bootstrap-t CI, and the BCa bootstrap CI. The nominal confidence level was fixed at 95%. The simulation design considered the parameter values $\theta \in$ {0.2, 0.3, 0.5, 0.75, 1, 1.5, 2, 2.5, 5, 10} and the sample sizes $n \in$ {10, 20, 30, 50, 100, 200}.

For each combination of θ and n, independent random samples were generated from the IXG distribution, and the four CIs were constructed. For each Monte Carlo sample, B=2000 bootstrap samples were used to construct the bootstrap-t and BCa bootstrap CIs. Their performance was evaluated using the empirical coverage probability (ECP) and average width (AW). The ECP was defined as the proportion of CIs containing the true parameter value, whereas the AW was calculated as the average difference between the upper and lower confidence limits over all Monte Carlo replications. A satisfactory CI should attain an ECP close to the nominal confidence level of 0.95 while maintaining a relatively small AW.

The simulation results are presented in Tables 1 and 2. Overall, the likelihood-based and Wald-type CIs exhibited the most reliable coverage performance. Across nearly all combinations of the parameter value and sample size, their ECPs remained close to the nominal confidence level of 0.95. In contrast, the bootstrap-t and BCa bootstrap CIs generally exhibited undercoverage when the sample size was small. However, the coverage performance of both bootstrap procedures improved steadily as n increased.

For θ=0.2 and n=10, the likelihood-based and Wald-type CIs produced ECPs of 0.9480 and 0.9485, respectively. In comparison, the bootstrap-t and BCa bootstrap intervals yielded noticeably lower ECPs of 0.8935 and 0.8825. When the sample size increased to n=200, the corresponding ECPs were 0.9545, 0.9535, 0.9525, and 0.9455, indicating substantial improvement in the performance of the bootstrap procedures. A similar pattern was observed for θ=0.3, where the bootstrap-t and BCa bootstrap ECPs increased from 0.8850 and 0.8805 at n=10 to 0.9485 and 0.9510 at n=200, respectively.

For moderate parameter values, the likelihood-based and Wald-type CIs continued to provide stable coverage. At θ=0.5 and n=10, their ECPs were 0.9505 and 0.9530, whereas those of the bootstrap-t and BCa bootstrap intervals were 0.8905 and 0.8785. At n=200 all four methods performed considerably better, with ECPs ranging from 0.9435 to 0.9515. For θ=0.75 the Wald-type interval attained an ECP of 0.9590 at n=10, whereas the likelihood-based CI attained 0.9500. The corresponding bootstrap-t and BCa bootstrap ECPs were 0.8970 and 0.8815. At n=200, however, the ECPs of all four methods were much closer to the target level, ranging from 0.9475 to 0.9550.

Table 1 Empirical coverage probability and average width of the 95% CIs for the parameter of the IXG distribution (θ = 0.2, 0.3, 0.5, 0.75, 1).

θ	n	Empirical Coverage Probability				Average Width			
		Likelihood	Wald	Boot-t	BCa	Likelihood	Wald	Boot-t	BCa
0.2	10	0.9480	0.9485	0.8935	0.8825	0.1571	0.1564	0.1405	0.1486
	20	0.9525	0.9580	0.9265	0.9275	0.1093	0.1091	0.1030	0.1050
	30	0.9555	0.9565	0.9355	0.9325	0.0880	0.0879	0.0835	0.0846
	50	0.9420	0.9440	0.9270	0.9320	0.0678	0.0677	0.0660	0.0665
	100	0.9455	0.9445	0.9415	0.9440	0.0477	0.0477	0.0467	0.0470
	200	0.9545	0.9535	0.9525	0.9455	0.0336	0.0336	0.0333	0.0335
0.3	10	0.9455	0.9505	0.8850	0.8805	0.2436	0.2425	0.2150	0.2284
	20	0.9550	0.9550	0.9250	0.9195	0.1663	0.1660	0.1562	0.1594
	30	0.9625	0.9640	0.9435	0.9420	0.1347	0.1345	0.1292	0.1310
	50	0.9555	0.9550	0.9445	0.9445	0.1040	0.1039	0.1015	0.1023
	100	0.9495	0.9480	0.9435	0.9430	0.0731	0.0731	0.0719	0.0723
	200	0.9520	0.9525	0.9485	0.9510	0.0516	0.0515	0.0508	0.0511
0.5	10	0.9505	0.9530	0.8905	0.8785	0.4232	0.4212	0.3696	0.3958
	20	0.9495	0.9565	0.9190	0.9090	0.2909	0.2902	0.2716	0.2787
	30	0.9590	0.9600	0.9365	0.9335	0.2337	0.2333	0.2244	0.2274
	50	0.9415	0.9435	0.9365	0.9360	0.1792	0.1791	0.1748	0.1763
	100	0.9525	0.9540	0.9470	0.9470	0.1266	0.1266	0.1247	0.1253
	200	0.9475	0.9515	0.9435	0.9470	0.0893	0.0893	0.0883	0.0887
0.75	10	0.9500	0.9590	0.8970	0.8815	0.6659	0.6623	0.5855	0.6380
	20	0.9370	0.9455	0.9105	0.9115	0.4543	0.4530	0.4240	0.4364
	30	0.9475	0.9535	0.9255	0.9260	0.3653	0.3647	0.3484	0.3541
	50	0.9505	0.9500	0.9340	0.9335	0.2800	0.2797	0.2708	0.2736
	100	0.9460	0.9440	0.9380	0.9395	0.1975	0.1974	0.1940	0.1953
	200	0.9550	0.9545	0.9475	0.9515	0.1390	0.1390	0.1374	0.1383
1	10	0.9415	0.9490	0.8840	0.8760	0.9276	0.9219	0.7977	0.8728
	20	0.9595	0.9630	0.9290	0.9240	0.6253	0.6234	0.5796	0.5981
	30	0.9410	0.9505	0.9230	0.9190	0.5054	0.5044	0.4793	0.4874
	50	0.9480	0.9525	0.9380	0.9385	0.3874	0.3870	0.3751	0.3792
	100	0.9440	0.9465	0.9340	0.9390	0.2716	0.2715	0.2654	0.2676
	200	0.9535	0.9560	0.9510	0.9465	0.1920	0.1919	0.1894	0.1904

The same overall conclusions were obtained for the larger parameter values reported in Table 2. For θ=1 and n=10, the likelihood-based and Wald-type CIs achieved ECPs of 0.9415 and 0.9490, compared with 0.8840 and 0.8760 for the bootstrap-t and BCa CIs. At n=200, the ECPs were 0.9535, 0.9560, 0.9510, and 0.9465, respectively. For θ=1.5, the two asymptotic intervals again maintained coverage close to 0.95 over all

sample sizes, whereas the two bootstrap intervals showed appreciable undercoverage for $n \leq 50$. At n=10, both bootstrap methods produced an ECP of 0.8815; by n=200, their ECPs had increased to 0.9445 and 0.9415.

For θ=2 the Wald-type interval attained relatively high ECPs for small samples, including 0.9650 at n=10, whereas the likelihood-based CI attained 0.9520. Although this indicates slight overcoverage by the Wald-type CI in this setting, both methods remained substantially closer to the nominal level than the bootstrap-t and BCa bootstrap CIs, whose ECPs were 0.8865 and 0.8695. At n=200, all methods achieved satisfactory coverage, with ECPs between 0.9500 and 0.9575. For θ=2.5, the same improvement with increasing sample size was observed. The bootstrap-t and BCa bootstrap ECPs increased from 0.8755 and 0.8735 at $n=10$ to 0.9535 and 0.9540 at n=200, respectively.

For θ=5 and θ=10, the likelihood-based and Wald-type CIs consistently provided ECPs close to the nominal level of 0.95 for all sample sizes. However, the both bootstrap-based CIs showed lower ECPs for small samples, with improved performance as n increased. For example, when θ=10 and n=10, the ECPs were 0.9450, 0.9570, 0.8770, and 0.8715 for the likelihood-based, Wald-type, bootstrap-t, and BCa bootstrap CIs, respectively. At n=200, the corresponding ECPs increased to 0.9520, 0.9540, 0.9490, and 0.9515.

With respect to interval width, all four methods produced progressively narrower intervals as the sample size increased. This pattern was consistent across all parameter settings and reflects the increasing precision of parameter estimation with larger samples. In addition, the AW increased with θ for a fixed sample size. For example, at n=10 the AW of the likelihood-based interval increased from 0.1571 for θ=0.2 to 2.6725 for θ=2.5. Similarly, the AW of the Wald-type CI increased from 0.1564 to 2.6516 over the same parameter range.

The likelihood-based and Wald-type CIs had almost identical widths in most scenarios. The Wald-type interval was usually marginally narrower than the likelihood-based interval, particularly for small and moderate sample sizes. For instance, when θ=1 and n=10, their AWs were 0.9276 and 0.9219, respectively. This difference decreased as the sample size increased; at n=200, the corresponding AWs were 0.1920 and 0.1919. This close agreement between the two methods is consistent with their common reliance on the large-sample behavior of the ML estimator.

The bootstrap-t CI generally yielded the smallest AW among the four CI methods. The reduction was particularly evident for small sample sizes. At θ=0.5 and n=10 for example, its AW was 0.3696, compared with 0.4232, 0.4212, and 0.3958 for the likelihood-based, Wald-type, and BCa bootstrap CIs, respectively. Similarly, for θ=2.5 and n=10 the bootstrap-t CI had an AW of 2.2404, whereas the corresponding values for the likelihood-based, Wald-type, and BCa bootstrap CIs were 2.6725, 2.6516, and 2.5179. However, the shorter widths of the bootstrap-t CIs were accompanied by substantial undercoverage, particularly when n was small. Therefore, the apparent gain in precision was achieved at the expense of inadequate coverage.

Table 2 Empirical coverage probability and average width of the 95% CIs for the parameter of the IXG distribution (θ = 1.5, 2, 2.5, 5, 10).

θ	n	Empirical Coverage Probability				Average Width			
		Likelihood	Wald	Boot-t	BCa	Likelihood	Wald	Boot-t	BCa
1.5	10	0.9435	0.9580	0.8815	0.8815	1.4766	1.4664	1.2556	1.3947
	20	0.9505	0.9580	0.9115	0.9125	0.9944	0.9909	0.9232	0.9553
	30	0.9435	0.9440	0.9255	0.9285	0.7984	0.7966	0.7591	0.7752
	50	0.9470	0.9445	0.9220	0.9245	0.6140	0.6132	0.5903	0.5979
	100	0.9500	0.9485	0.9370	0.9385	0.4286	0.4283	0.4180	0.4211
	200	0.9485	0.9460	0.9445	0.9415	0.3033	0.3032	0.2996	0.3010
2	10	0.9520	0.9650	0.8865	0.8695	2.0615	2.0462	1.7457	1.9638
	20	0.9455	0.9520	0.9180	0.9110	1.3783	1.3730	1.2660	1.3123
	30	0.9400	0.9505	0.9130	0.9175	1.1218	1.1190	1.0525	1.0758
	50	0.9490	0.9530	0.9300	0.9320	0.8484	0.8472	0.8157	0.8261
	100	0.9415	0.9420	0.9355	0.9355	0.5949	0.5944	0.5860	0.5897
	200	0.9550	0.9570	0.9500	0.9575	0.4216	0.4215	0.4156	0.4180
2.5	10	0.9435	0.9520	0.8755	0.8735	2.6725	2.6516	2.2404	2.5179
	20	0.9460	0.9505	0.9095	0.9125	1.7932	1.7859	1.6527	1.7173
	30	0.9590	0.9595	0.9265	0.9295	1.4331	1.4293	1.3497	1.3775
	50	0.9460	0.9495	0.9305	0.9275	1.1028	1.1011	1.0606	1.0745
	100	0.9505	0.9465	0.9420	0.9440	0.7682	0.7676	0.7503	0.7562
	200	0.9540	0.9605	0.9535	0.9540	0.5452	0.5450	0.5385	0.5419
5	10	0.9465	0.9580	0.8795	0.8735	5.9765	5.9223	4.9236	5.6387
	20	0.9475	0.9540	0.9120	0.9110	3.9802	3.9610	3.6206	3.7823
	30	0.9480	0.9515	0.9225	0.9250	3.1861	3.1759	2.9509	3.0110
	50	0.9475	0.9495	0.9280	0.9285	2.4335	2.4290	2.3413	2.3725
	100	0.9480	0.9465	0.9385	0.9400	1.6951	1.6935	1.6663	1.6793
	200	0.9525	0.9545	0.9495	0.9510	1.2077	1.2072	1.1915	1.2018
10	10	0.9450	0.9570	0.8770	0.8715	13.3672	13.2296	10.8064	12.5827
	20	0.9470	0.9530	0.9110	0.9100	8.8531	8.8037	7.9678	8.3703
	30	0.9475	0.9510	0.9210	0.9240	7.0553	7.0289	6.4451	6.5720
	50	0.9480	0.9490	0.9270	0.9280	5.3829	5.3712	5.1814	5.2518
	100	0.9485	0.9460	0.9380	0.9395	3.7413	3.7373	3.6889	3.7194
	200	0.9520	0.9540	0.9490	0.9515	2.6760	2.6749	2.6393	2.6680

The BCa bootstrap CI was generally wider than the bootstrap-t CI but narrower than the likelihood-based and Wald-type CIs. Its coverage performance was often comparable to, but not systematically better than, that of the bootstrap-t CI. In several small-sample settings, the BCa bootstrap method exhibited the lowest coverage among the four methods. For example, at θ=2 and n=10 the BCa bootstrap CI produced an ECP of

0.8695, despite having an AW of 1.9638, which was wider than the bootstrap-t AW of 1.7457. This result indicates that the bias and acceleration corrections did not fully compensate for the small-sample skewness or estimation uncertainty in this setting.

As the sample size increased, the differences in AW among the four methods became considerably smaller. For θ=0.2 and n=200, the widths ranged only from 0.0333 to 0.0336. For θ=2.5 and n=200, the corresponding range was 0.5385—0.5452. At the same time, the ECPs of the bootstrap CIs approached the nominal confidence level. Therefore, the practical distinctions among the four methods became less pronounced for large samples.

In summary, the likelihood-based and Wald-type CIs provided the best overall balance between coverage accuracy and interval width. Their ECPs were consistently close to 0.95, even for relatively small samples, although the Wald-type CI occasionally displayed mild overcoverage. The likelihood-based method appeared slightly more stable across the complete range of parameter values and sample sizes, whereas the Wald-type method offered nearly identical performance with marginally shorter intervals in many cases. The bootstrap-t method generated the narrowest intervals but suffered from substantial undercoverage for small and moderate sample sizes. The BCa bootstrap CI partially reduced the difference in width between the bootstrap-t and asymptotic procedures but did not consistently resolve the undercoverage problem. Accordingly, the likelihood-based and Wald-type CIs are recommended when reliable coverage is the primary consideration. The bootstrap procedures may become competitive for large samples, where their ECPs approach the nominal level and their widths become comparable to those of the asymptotic intervals.

Applications to a Real Dataset

To illustrate the practical usefulness of the proposed CI methods, the IXG distribution was fitted to a real lifetime dataset originally reported by Lawless [14]. The dataset consists of the failure times, measured in minutes, of 15 electronic components subjected to an accelerated life test. Descriptive statistics for this dataset are shown in Table 3.

Table 3 Descriptive statistics for the failure times of electronic components subjected to an accelerated life test.

Statistic	Min	Q1	Median	Mean	Q3	Max	SD	Skew	Kurt
Value	1.40	11.45	22.20	27.55	41.80	66.20	20.7634	0.6310	2.1881

Observed failure times ranged from 1.40 to 66.20 minutes (median 22.20 min, mean 27.55 min). The first and third quartiles were 11.45 and 41.80 minutes respectively, showing a relatively wide spread in the observed failure times. The standard deviation was 20.7634 minutes, indicating a substantial variability between the observations. The positive skewness coefficient of 0.6310 indicates that the failure times distribution is moderately skewed to the right. Also, the kurtosis coefficient is 2.1881, indicating a relatively flat, or platykurtic, empirical distribution.

The ML estimate of the IXG distribution parameter was $\hat{\theta}$=11.10647, with an estimated standard error of 2.618884. The maximized log-likelihood value was -68.70466. To evaluate the relative fit of the IXG

distribution, it was compared with a broad collection of competing one-parameter lifetime distributions using the Akaike information criterion (AIC) [15] and Bayesian information criterion (BIC) [16]. Smaller values of AIC and BIC indicate a better balance between goodness of fit and model complexity. The AIC and BIC comparisons were included solely to assess the relative adequacy of the IXG distribution against competing lifetime distributions. These criteria were not used as performance measures in the simulation study, whose primary objective was to compare the finite-sample behavior of the four CI procedures.

Comparative analysis was conducted including twenty lifetime distributions. All the pdfs are defined for $x > 0$ and are characterized by a parameter $\theta > 0$:

1) Adya distribution [17]

$$f(x;\theta) = \frac{\theta^3}{\theta^4 + 2\theta^2 + 2}(\theta + x)^2 e^{-\theta x}$$

2) Akash distribution [18]

$$f(x;\theta) = \frac{\theta^3}{\theta^2 + 2}(1 + x^2)e^{-\theta x}$$

3) Akshaya distribution [19]

$$f(x;\theta) = \frac{\theta^4}{\theta^3 + 3\theta^2 + 6\theta + 6}(1 + x)^3 e^{-\theta x}$$

4) Amarendra distribution [20]

$$f(x;\theta) = \frac{\theta^4}{\theta^3 + \theta^2 + 2\theta + 6}(1 + x + x^2 + x^3)e^{-\theta x}$$

5) Aradhana distribution [21]

$$f(x;\theta) = \frac{\theta^3}{\theta^2 + 2\theta + 2}(1 + x)^2 e^{-\theta x}$$

6) Chris-Jerry distribution [22]

$$f(x;\theta) = \frac{\theta^2}{\theta + 2}(1 + \theta x^2)\, e^{-\theta x}$$

7) Double XShanker distribution [23]

$$f(x;\theta) = \frac{\theta^2}{(\theta^2 + 1)^3}\left(\theta^5 + 3\theta^3 + 3\theta + x\right) e^{-\theta x}$$

8) Exponential distribution

$$f(x;\theta) = \theta e^{-\theta x}$$

9) Fav-Jerry distribution [24]

$$f(x;\theta) = \frac{\theta}{\theta^2 + 2}\left(2 + \theta^3 x\right) e^{-\theta x}$$

10) Garima distribution [25]

$$f(x;\theta) = \frac{\theta}{\theta + 2}(1 + \theta + \theta x)e^{-\theta x}$$

11) Haq distribution [26]

$$f(x;\theta) = \left(\frac{\theta}{1 + \theta}\right)^2 \left(2 + \theta + \frac{\theta x^2}{2}\right) e^{-\theta x}$$

12) Ishita distribution [27]

$$f(x;\theta)=\frac{\theta^3}{\theta^3+2}(\theta+x^2)e^{-\theta x}$$

13) Iwok-Nwikpe distribution [28]

$$f(x;\theta)=\frac{\theta^3}{\theta+2}(x^2+x)e^{-\theta x}$$

14) Komal distribution [29]

$$f(x;\theta)=\frac{\theta^2}{\theta^2+\theta+1}(1+\theta+x)e^{-\theta x}$$

15) Lindley distribution [30]

$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$$

16) Nwikpe distribution [31]

$$f(x;\theta)=\frac{\theta^3}{12(\theta^2+10)}\left(\theta^3x^5+12\right)e^{-\theta x}$$

17) Pratibha distribution [32]

$$f(x;\theta)=\frac{\theta^3}{\theta^3+\theta+2}(\theta+x+x^2)e^{-\theta x}$$

18) Rama distribution [33]

$$f(x;\theta)=\frac{\theta^4}{\theta^3+6}(1+x^3)e^{-\theta x}$$

19) Shanker distribution [34]

$$f(x;\theta)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}$$

20) Sujatha distribution [35]

$$f(x;\theta)=\frac{\theta^3}{\theta^2+\theta+2}\left(1+x+x^2\right)e^{-\theta x}$$

The results in Table 4 show that the IXG distribution produced AIC and BIC values of 139.4093 and 140.1174, respectively. These were substantially lower than those obtained from all competing distributions. For example, the Fav-Jerry distribution, which provided the next smallest AIC value among the distributions considered, yielded an AIC value of 673.9954, with a corresponding BIC value of 674.7035. Therefore, according to both information criteria, the IXG distribution provided the best fit to the dataset among all candidate distributions considered. Although AIC and BIC provide evidence on the relative fit of the IXG distribution compared with alternative distributions, they do not by themselves assess absolute goodness of fit. Therefore, the adequacy of the fitted IXG distribution was further examined using the Kolmogorov-Smirnov test and the P-P plot.

The adequacy of the fitted IXG distribution was further examined using the one-sample Kolmogorov-Smirnov test [36]. The resulting test statistic and p-value were D=0.25066 and p-value=0.2565, respectively. Because the p-value exceeded the conventional significance level of 0.05, the null hypothesis that the

observations follow the fitted IXG distribution was not rejected. This result provides no evidence of a statistically significant discrepancy between the empirical distribution of the data and the fitted IXG distribution.

A probability-probability (P-P) plot was also constructed to assess the goodness of fit graphically. The empirical probabilities were plotted against the theoretical probabilities obtained from the fitted IXG distribution. As shown in Figure 2, the plotted points generally followed the 45-degree reference line, although some departures were visible in the central and upper portions of the plot. Overall, the pattern indicates reasonable agreement between the empirical and fitted probabilities and supports the conclusion obtained from the Kolmogorov-Smirnov test.

After establishing the suitability of the IXG distribution, four 95% CIs for θ were constructed using the likelihood-based, Wald-type, bootstrap-t, and BCa bootstrap procedures. For both bootstrap methods, 2,000 bootstrap replications were used. The resulting CIs are presented in Table 5.

Table 4. Evaluation of model fit for various distributions used for the waiting time of electronic-component failure-time data, with models ranked in ascending order based on their AIC values.

Distribution	Estimate (SE)	LogLik	AIC	BIC	Rank(AIC)
Inverse Xgmama	11.1065 (2.6189)	-68.7047	**139.4093**	**140.1174**	1
Fav-Jerry	0.0363 (0.0094)	-335.9977	673.9954	674.7035	2
Exponential	0.0363 (0.0094)	-335.9977	673.9955	674.7035	3
Garima	0.0546 (0.0123)	-358.1288	718.2577	718.9657	4
Haq	0.0954 (0.0150)	-419.8184	841.6368	842.3449	5
Lindley	0.0725 (0.0132)	-474.0637	950.1273	950.8354	6
Komal	0.0701 (0.0128)	-476.7856	955.5712	956.2793	7
Chris-Jerry	0.1034 (0.0158)	-519.6413	1041.2830	1041.9910	8
Double XShanker	0.0722 (0.0131)	-535.3163	1072.6330	1073.3410	9
Shanker	0.0725 (0.0132)	-558.5101	1119.0200	1119.7280	10
Aradhana	0.1051 (0.0157)	-635.2115	1272.4230	1273.1310	11
Sujatha	0.1067 (0.0159)	-646.9993	1295.9990	1296.7070	12
Akash	0.1085 (0.0161)	-664.3922	1330.7840	1331.4920	13
Pratibha	0.1071 (0.0160)	-713.0125	1428.0250	1428.7330	14
Nwikpe	0.2072 (0.0228)	-721.3043	1444.6090	1445.3170	15
Iwok-Nwikpe	0.1071 (0.0160)	-737.4155	1476.8310	1477.5390	16
Ishita	0.1090 (0.0162)	-762.5840	1527.1680	1527.8760	17
Adya	0.1088 (0.0162)	-782.9293	1567.8590	1568.5670	18
Akshaya	0.1401 (0.0181)	-796.5154	1595.0310	1595.7390	19
Amarendra	0.1433 (0.0185)	-827.1849	1656.3700	1657.0780	20
Rama	0.1452 (0.0187)	-857.9366	1717.8730	1718.5810	21

The best-fitting model consistent with the least AIC and BIC criteria is specified by bold values.

All four intervals contained the ML estimate, but their widths differed appreciably. The likelihood-based and Wald-type intervals were relatively similar, with widths of 10.3366 and 10.2658, respectively. The Wald-type interval was marginally narrower than the likelihood-based interval, which is consistent with the close agreement between these two procedures observed in the simulation study.

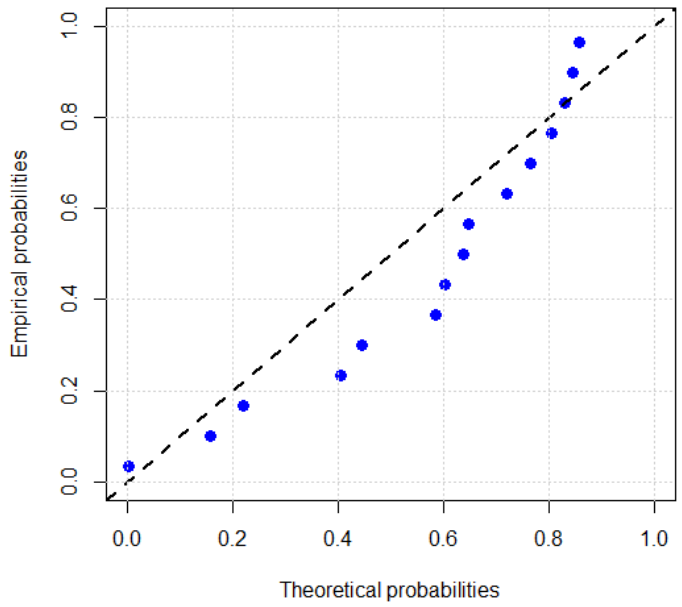


Figure 2 P–P plot of the fitted IXG distribution for the electronic-component failure-time data.

In contrast, the bootstrap-t and BCa bootstrap CIs were considerably wider. The bootstrap-t CI had a width of 15.0067, whereas the BCa bootstrap CI had the largest width, equal to 15.7690. The greater widths of the bootstrap intervals reflect the additional finite-sample uncertainty captured through resampling. In particular, the pronounced asymmetry of the bootstrap limits, especially toward the upper endpoint, suggests that the sampling distribution of the estimator may be positively skewed for this small dataset.

Table 5 The 95% confidence limits and interval widths for the IXG parameter based on the electronic-component failure-time data.

Method	Lower limit	Upper limit	Width
Likelihood-based	6.7661	17.1027	10.3366
Wald-type	5.9735	16.2394	10.2658
Bootstrap-t	5.0824	20.0891	15.0067
BCa bootstrap	4.8558	20.6248	15.7690

The likelihood-based interval was slightly asymmetric around the ML estimate, with lower and upper limits of 6.7661 and 17.1027. This asymmetry arises naturally because the interval is obtained directly from the shape of the likelihood function rather than from a symmetric normal approximation. By comparison, the Wald-type interval was approximately symmetric around $\hat{\theta}$, as expected from its construction using the asymptotic normal distribution of the ML estimator.

The bootstrap-t and BCa bootstrap CIs extended further in both directions than the two likelihood-based procedures. Among the four methods, the BCa bootstrap CI provided the most conservative assessment of parameter uncertainty. Its lower endpoint was 4.8558 and its upper endpoint was 20.6248. The difference between the bootstrap and asymptotic intervals is not unexpected because the dataset contains only 15 observations. For such a small sample, the finite-sample distribution of the ML estimator may differ substantially from its asymptotic normal approximation.

Overall, the AIC and BIC, Kolmogorov-Smirnov test, and P-P plot provide consistent evidence that the IXG distribution is an appropriate model for the data. The likelihood-based and Wald-type methods yielded relatively concise CIs, whereas the bootstrap-t and BCa bootstrap methods reflected greater uncertainty through wider limits. Considering the favorable coverage performance observed in the simulation study, the likelihood-based and Wald-type CIs appear to provide an effective balance between precision and reliability for this application. Nevertheless, the wider bootstrap intervals highlight the uncertainty associated with parameter estimation from a small sample and should be considered when conservative inference is desired.

Discussion and Conclusions

The simulation findings provide several insights into the finite-sample behavior of CIs for the parameter of the IXG distribution. The most important result is the consistently reliable performance of the likelihood-based and Wald-type CIs in terms of ECP. Across the parameter values and sample sizes considered, both methods generally attained ECPs close to the nominal level of 0.95. This result indicates that the large-sample approximations underlying these methods remain reasonably accurate even for relatively small samples under the simulation settings examined.

The likelihood-based CI showed particularly stable coverage across the full range of parameter values. Its satisfactory performance may be attributed to the fact that it is constructed directly from the relative likelihood and therefore preserves more information about the shape and asymmetry of the likelihood function than an interval based only on the local shape of the likelihood function. This feature can be important for the IXG distribution because its likelihood may be asymmetric in small samples, especially when the parameter value is relatively large. In contrast, the Wald-type interval relies on a quadratic approximation to the log-likelihood function in a neighborhood of the ML estimate. Despite this approximation, the Wald-type interval performed almost identically to the likelihood-based interval in most cases and frequently produced slightly smaller average widths.

The strong performance of the Wald-type interval suggests that the observed information provided an adequate estimate of the sampling variability of the ML estimator for the settings considered. However, slight overcoverage was observed in some small-sample settings, particularly for larger parameter values. This behavior may arise from finite-sample deviations from the asymptotic normality of the ML estimator or from conservative estimation of its standard error. Although the degree of overcoverage was generally modest, the result indicates that the nominal normal approximation is not uniformly exact across all parameter settings.

The bootstrap-t CI produced the shortest average widths in nearly all scenarios. At first sight, this finding suggests greater precision. However, its narrower intervals were accompanied by substantial undercoverage for small and moderate sample sizes. Therefore, the reduction in width should not be interpreted

as an unqualified improvement in inferential efficiency. A CI that is substantially narrower than competing intervals but fails to achieve the nominal coverage probability may provide an overly optimistic representation of parameter uncertainty. In the present setting, the bootstrap-t method appears to underestimate the uncertainty associated with the ML estimator when the available sample is limited.

Several factors may explain the small-sample undercoverage of the bootstrap-t interval. First, its performance depends on the quality of the estimated standard error within each bootstrap sample. For small samples, the observed information can vary considerably across bootstrap replications, resulting in an unstable studentized statistic. Second, the IXG distribution is positively skewed and heavy-tailed, and ordinary positive moments do not exist. These characteristics may induce substantial skewness in the finite-sample distribution of the ML estimator and slow the convergence of the studentized statistic to its limiting distribution. Third, the numerical estimation of the parameter in each bootstrap replication may introduce additional variability, particularly when bootstrap samples contain extreme observations.

The BCa bootstrap CI was designed to adjust percentile endpoints for median bias and skewness through the bias-correction and acceleration constants. However, the simulation results showed that this adjustment did not consistently eliminate the undercoverage observed for small samples. In many cases, the BCa bootstrap CI was wider than the bootstrap-t interval but still had a coverage probability below the nominal level. Therefore, the additional width introduced by the BCa correction did not always translate into a sufficient improvement in coverage accuracy.

One possible explanation is that the jackknife-based estimate of the acceleration constant may be unstable when the sample size is small or when individual observations exert considerable influence on the ML estimate. This issue is particularly relevant for heavy-tailed lifetime distributions, for which deleting an extreme observation can lead to a relatively large change in the estimate. Moreover, although the BCa bootstrap method corrects for median bias and skewness, it does not necessarily account fully for other finite-sample irregularities, such as substantial tail imbalance or instability in numerical optimization. The results therefore demonstrate that higher-order bootstrap corrections do not automatically guarantee superior finite-sample performance for every parametric model.

The observed increase in average width with the parameter value is consistent with the changing scale and variability of the IXG distribution. For a fixed sample size, larger values of θ were associated with wider CIs for all four methods. This pattern indicates that the absolute uncertainty in estimating θ increased over the parameter range considered. However, the widths decreased markedly as the sample size increased, which is consistent with the expected improvement in estimation precision and the consistency of the ML estimator.

Another notable finding is the convergence of the four CI procedures as the sample size increased. For n=100 and, more clearly, for n=200, the differences in both ECP and AW became relatively small. The bootstrap procedures attained coverage probabilities much closer to the nominal level, while their widths approached those of the likelihood-based and Wald-type CIs. This convergence supports the asymptotic validity of all four procedures and suggests that the principal differences among them are primarily finite-sample phenomena.

From a practical perspective, the choice of CI should be guided by the relative importance assigned to coverage accuracy and interval width. When reliable coverage is the primary objective, the likelihood-based

and Wald-type intervals are preferable, particularly for small and moderate sample sizes. Between these two methods, the likelihood-based interval may be favored when asymmetry in the likelihood is a concern, whereas the Wald-type interval offers a simpler computational form and nearly identical performance in most settings. The bootstrap-t interval may be attractive because of its shorter width, but its use with small samples should be approached cautiously because the associated undercoverage can be substantial. The BCa bootstrap CI provides some correction for bias and skewness but does not consistently outperform the bootstrap-t CI in terms of coverage.

The findings also have implications for applications involving lifetime, reliability, and survival data. Such datasets are frequently limited in size, and parameter uncertainty can have direct consequences for reliability measures, risk assessment, and survival probabilities. In these circumstances, selecting an interval solely because it is narrow may lead to misleadingly precise conclusions. The present results suggest that intervals with dependable coverage should be prioritized over those offering modest reductions in width.

The conclusions of this simulation study are conditional on the parameter values, sample sizes, confidence level, and bootstrap implementation considered. Further investigation could examine other nominal confidence levels, larger parameter ranges, censored samples, contaminated data, and model misspecification. It would also be useful to investigate transformed Wald and bootstrap CIs, such as intervals constructed on the logarithmic scale, because such transformations automatically respect the restriction $\theta > 0$ and may reduce the effects of skewness. Additional comparisons with profile-likelihood, modified likelihood-root [37], Bayesian credible, and double-bootstrap intervals could provide a broader assessment of inferential procedures for the IXG parameter.

The real-data application further supported the practical usefulness of the proposed procedures. For the electronic-component failure-time data, the IXG distribution provided the smallest AIC and BIC values among the competing models, while the Kolmogorov-Smirnov test and the P-P plot indicated an adequate fit. The likelihood-based and Wald-type CIs were also relatively close in both limits and width, whereas the bootstrap-t and BCa bootstrap CIs were wider and more asymmetric. These findings are consistent with the simulation results, in which the likelihood-based and Wald-type procedures generally offered a more favorable balance between coverage accuracy and interval width, particularly for small samples.

Overall, the results indicate that no single method simultaneously minimized interval width and achieved uniformly accurate coverage in every finite-sample setting. The likelihood-based and Wald-type CIs provided the most satisfactory compromise between coverage accuracy and precision. The bootstrap-t and BCa bootstrap CIs became competitive only as the sample size increased. Therefore, for routine inference on the parameter of the IXG distribution, the likelihood-based and Wald-type methods are recommended, especially when the sample size is small or moderate.

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