Research Article

On Algorithms for Computing Derivations and Antiderivations of Leibniz Algebras

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ABSTRACT

The concepts of antiderivation and biderivation were first introduced for right Leibniz algebras in 1993. In this paper, we extended these definitions to left Leibniz algebras and developed Maple software programs specifically for computing derivations and antiderivations of Leibniz algebras. As an application, we provided a complete classification of biderivations for non-Lie Leibniz algebras of dimensions up to three over the complex field.

Keywords: Leibniz algebra, Derivation, Antiderivation, Biderivation, Algorithm

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Introduction

The concept of biderivations, initially introduced by Brešar [1] in 1995 for associative rings and extended to Lie algebras by Wang, Yu, and Chen [2] in 2011, has garnered significant attention due to its broad applicability across various fields [3]. Leibniz algebras, first explored by Bloh [4] in 1965 and later introduced by Loday [5] in 1993 as a generalization of Lie algebras, have been extensively studied since then [6-10]. This broader framework allows Leibniz algebras to model structures in various mathematical and physical contexts, such as noncommutative geometry, deformation theory, and string theory. Their rich structure has also found applications in homology theory and the study of algebraic operads. However, in the literature, some of these results are proved for left Leibniz algebras and some are proved for right Leibniz algebras. Loday [5] provided definitions of derivation, antiderivation, and biderivation for right Leibniz algebras. Recent work by Mancini [9] in 2023 classified low-dimensional right Leibniz algebras with a focus on biderivations. In this paper, following Barnes [6], we focus on left Leibniz algebras and investigate their derivations, antiderivations, and biderivations. We begin by revisiting fundamental concepts of Leibniz algebras in the Preliminaries, delineating the notion of derivation, antiderivation, and biderivation for right Leibniz algebras. We then extend these definitions to left Leibniz algebras, establish their basic properties, and present Maple software algorithms for computing derivations and antiderivations. Finally, in the last section, we utilize these algorithms to classify biderivations in two- and three-dimensional non-Lie left Leibniz algebras over the complex field, using the classification given by Demir, Misra, and Stitzinger [7].

Preliminaries

Let $\mathbb{F}$ be an algebraically closed field with characteristic zero. A (left) *Leibniz algebra* L is an $\mathbb{F}$-vector space equipped with bracket $[\ ,\ ] : L\times L\to L$ such that

1. $[\ ,\ ]$ is $\mathbb{F}$-bilinear, i.e., $[\alpha x+\beta y,z]=\alpha[x,z]+\beta[y,z]$ and $[x,\alpha y+\beta z]=\alpha[x,y]+\beta[x,z]$ for all $x,y,z\in L,\ \alpha,\beta\in\mathbb{F}$.
2. It satisfies the Leibniz identity: $[x,[y,z]]=[[x,y],z]+[y,[x,z]]$ for all $x,y,z\in L$.

When the bracket satisfies $[x,x]=0$ for all $x\in L$, the Leibniz identity becomes the Jacobi identity $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ for all $x,y,z\in L$. Thus, the Leibniz algebra can be viewed as a generalization of the Lie algebra.

A *derivation* of a Leibniz algebra L is an $\mathbb{F}$-linear map $d : L\to L$ satisfying

$$d([x,y])=[d(x),y]+[x,d(y)] \text{ for all } x,y\in L.$$

The set of all derivations of a Leibniz algebra L is denoted by $\mathrm{Der}(L)$. It is known that $\mathrm{Der}(L)$ is a Lie algebra with the commutator bracket $[d_1,d_2]=d_1\circ d_2-d_2\circ d_1$ for all $d_1,d_2\in\mathrm{Der}(L)$.

Example 1 Let L be a left Leibniz algebra and $x\in L$. Define the *left multiplication operator* $L_x : L\to L$ by $L_x(y)=[x,y]$ for all $y\in L$. Then for all $y,z\in L$, we have

$$L_x([y,z])=[x,[y,z]]=[[x,y],z]+[y,[x,z]]=[L_x(y),z]+[y,L_x(z)].$$

Thus, L_x is a derivation.

Let $x \in L$. Define the *right multiplication operator* $R_x : L \to L$ by $R_x(y) = [y, x]$ for all $y \in L$. A right Leibniz algebra is defined as a vector space equipped with a bilinear multiplication such that the right multiplication operator R_x is a derivation. Throughout this paper, unless otherwise stated, the term 'Leibniz algebra' specifically refers to the left Leibniz algebra. As illustrated by the following example, a left Leibniz algebra is not necessarily a right Leibniz algebra.

Example 2 Let L be a 2-dimensional vector space with the basis $\{x, y\}$. Define

$$[x,x]=0,\ [x,y]=0,\ [y,x]=x,\ [y,y]=x,$$

and extend this to all elements in L by linearity. Note that L is not a Lie algebra because $[y,y]=x \neq 0$. With this bracket structure, L forms a left Leibniz algebra. However, since $[[y,y],y]=[x,y]=0$ but $[y,[y,y]]+[[y,y],y]=[y,x]+[x,y]=x$, it follows that $[[y,y],y] \neq [y,[y,y]]+[[y,y],y]$. Hence, L is not a right Leibniz algebra.

Let M and N be two subspaces of a Leibniz algebra L. Then we denote the subspace $\mathrm{span}_{\mathbb{F}}\{[x,y] \| x \in M,\ y \in N\}$ by $[M,N]$. A subspace I of a Leibniz algebra L is an *ideal* of L if $[L,I] \subseteq I$ and $[I,L] \subseteq I$. Given any Leibniz algebra L, we denote $\mathrm{Leib}(L) = \mathrm{span}_{\mathbb{F}}\{[x,x] \| x \in L\}$. It is known that $\mathrm{Leib}(L)$ is an abelian ideal of L [7].

A *module* of a Lie algebra L is a vector space V with an operation $\cdot : L \times V \to V$ such that for all $x, y \in L$, $u, v \in V$, and $a, b \in \mathbb{F}$,

1. $x \cdot (au + bv) = a(x \cdot u) + b(x \cdot v)$,
2. $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$,
3. $[x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.

Loday [5] introduced the notions of antiderivation and biderivation for right Leibniz algebras as follows.

An *antiderivation* of a right Leibniz algebra L is an $\mathbb{F}$-linear map $D : L \to L$ such that

$$D([x,y]) = [D(x), y] - [D(y), x] \text{ for all } x, y \in L.$$

The set of all antiderivations of a Leibniz algebra L is denoted by $\mathrm{Antider}(L)$.

A *biderivation* of a right Leibniz algebra L is a pair (d, D) where d is a derivation and D is an antiderivation, such that

$$[x, d(y)] = [x, D(y)] \text{ for all } x, y \in L.$$

The set of all biderivations of a Leibniz algebra L is denoted by $\mathrm{Bider}(L)$.

In [9], Mancini showed that $\mathrm{Antider}(L)$ has a $\mathrm{Der}(L)$-module structure with the operation

$$d \cdot D = [D, d] = D \circ d - d \circ D$$

for all $d \in \mathrm{Der}(L)$ and $D \in \mathrm{Antider}(L)$. He also showed that $\mathrm{Bider}(L)$ has a Leibniz algebra structure with the bracket $[\ ,\ ] : \mathrm{Bider}(L) \times \mathrm{Bider}(L) \to \mathrm{Bider}(L)$ by

$$[(d,D), (d',D')] = (d \circ d' - d' \circ d, D \circ d' - d' \circ D)$$

for all $d, d' \in \mathrm{Der}(L)$ and $D, D' \in \mathrm{Antider}(L)$.

Antiderivations and biderivations

Motivated by results for right Leibniz algebras in [9], we define antiderivation and biderivation of left Leibniz algebras and study their properties. Note that the definition of derivation of left Leibniz algebras is the same as in the case of right Leibniz algebras.

Definition 1 An *antiderivation* of a left Leibniz algebra L is an $\mathbb{F}$-linear map $D : L \to L$ such that

$$D([x, y]) = [x, D(y)] - [y, D(x)] \text{ for all } x, y \in L.$$

The set of all antiderivations of a left Leibniz algebra L is also denoted by $\mathrm{Antider}(L)$.

Example 3 Let L be a left Leibniz algebra and $x \in L$. Consider the right multiplication operator $R_x : L \to L$ by $R_x(y) = [y, x]$ for all $y \in L$. Then for all $y, z \in L$, we have

$$R_x([y,z]) = [[y,z],x] = [y,[z,x]] - [z,[y,x]] = [y, R_x(z)] - [z, R_x(y)].$$

Thus, R_x is an antiderivation.

Proposition 1 The set of antiderivations of a left Leibniz algebra L has a $\mathrm{Der}(L)$-module structure with the operation

$$d \cdot D = [d, D] = d \circ D - D \circ d$$

for all $d \in \mathrm{Der}(L)$ and $D \in \mathrm{Antider}(L)$.

Proof. Let $d, d' \in \mathrm{Der}(L)$, $D, D' \in \mathrm{Antider}(L)$, and $\alpha, \beta \in \mathbb{F}$. Then

$$\begin{aligned}
(d \circ D - D \circ d)([x,y]) &= d(D([x,y])) - D(d([x,y])) \\
&= d([x, D(y)] - [y, D(x)]) - D([d(x), y] + [x, d(y)]) \\
&= [d(x), D(y)] + [x, d(D(y))] - [d(y), D(x)] - [y, d(D(x))] \\
&\quad - [d(x), D(y)] + [y, D(d(x))] - [x, D(d(y))] + [d(y), D(x)] \\
&= [x, d(D(y))] - [y, d(D(x))] + [y, D(d(x))] - [x, D(d(y))] \\
&= [x, (d \circ D - D \circ d)(y)] - [y, (d \circ D - D \circ d)(x)].
\end{aligned}$$

Thus, $d \circ D - D \circ d \in \mathrm{Antider}(L)$. Also,

$$\begin{aligned}
d \cdot (\alpha D + \beta D') &= [d, \alpha D + \beta D'] = d \circ (\alpha D + \beta D') - (\alpha D + \beta D') \circ d \\
&= \alpha[d, D] + \beta[d, D'] = \alpha(d \cdot D) + \beta(d \cdot D'), \\
(\alpha d + \beta d') \cdot D &= [\alpha d + \beta d', D] = \alpha[d, D] + \beta[d', D] = \alpha(d \cdot D) + \beta(d' \cdot D), \\
[d, d'] \cdot D &= [[d, d'], D] = [d \circ d' - d' \circ d, D] = [d \circ d', D] - [d' \circ d, D] \\
&= [d, [d', D]] - [d', [d, D]] = d \cdot (d' \cdot D) - d' \cdot (d \cdot D).
\end{aligned}$$

Hence, $\mathrm{Antider}(L)$ is a $\mathrm{Der}(L)$-module.

Definition 2 Let L be a left Leibniz algebra. A *biderivation* of L is a pair

$$(d, D)$$

where d is a derivation and D is an antiderivation, such that

$$[d(x), y] = [D(x), y] \text{ for all } x, y \in L.$$

The set of all biderivations of a left Leibniz algebra L is also denoted by $\mathrm{Bider}(L)$.

Define the bracket $[\ ,\ ] : \mathrm{Bider}(L) \times \mathrm{Bider}(L) \to \mathrm{Bider}(L)$ by

$$\left[(d,D),(d',D')\right]=(d\circ d'-d'\circ d, d\circ D'-D'\circ d)$$

for all $(d,D),(d',D')\in \text{Bider}(L)$. Observe that for all $x,y\in L$,

$$\left[(d\circ d'-d'\circ d)(x),y\right]=\left[(d\circ D'-D'\circ d)(x),y\right]$$

as $\left[d'(x),y\right]=\left[D'(x),y\right]$. Then we have the following proposition.

Proposition 2 Bider(L) is a left Leibniz algebra.

Proof. Note that Bider(L) is a vector space over $\mathbb{F}$ since the bracket of L is $\mathbb{F}$-bilinear. Let $(d,D),(d',D'),(d'',D'')\in \text{Bider}(L)$ and $\alpha,\beta\in\mathbb{F}$. Clearly, $d\circ d'-d'\circ d\in \text{Der}(L)$ and $d\circ D'-D'\circ d\in \text{Antider}(L)$. We also obtain that

$$\begin{aligned}\left[\alpha(d,D)+\beta(d',D'),(d'',D'')\right]&=\left[(\alpha d+\beta d',\alpha D+\beta D'),(d'',D'')\right]\\&=((\alpha d+\beta d')\circ d''-d''\circ(\alpha d+\beta d'),(\alpha d+\beta d')\circ D''-D''\circ(\alpha d+\beta d'))\\&=\alpha\left[(d,D),(d'',D'')\right]+\beta\left[(d',D'),(d'',D'')\right].\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left[(d,D),\alpha(d',D')+\beta(d'',D'')\right]&=\left[(d,D),(\alpha d'+\beta d'',\alpha D'+\beta D'')\right]\\&=(d\circ(\alpha d'+\beta d'')-(\alpha d'+\beta d'')\circ d,d\circ(\alpha D'+\beta D'')-(\alpha D'+\beta D'')\circ d)\\&=\alpha\left[(d,D),(d',D')\right]+\beta\left[(d,D),(d'',D'')\right].\end{aligned}$$

Hence, $[\ ,\ ]$ is bilinear. Consider

$$\begin{aligned}&\left[(d,D),\left[(d',D'),(d'',D'')\right]\right]\\&=\left[(d,D),(d'\circ d''-d''\circ d',d'\circ D''-D''\circ d')\right]\\&=(d\circ(d'\circ d''-d''\circ d')-(d'\circ d''-d''\circ d')\circ d,d\circ(d'\circ D''-D''\circ d')-(d'\circ D''-D''\circ d')\circ d)\\&=(d\circ d'\circ d''-d\circ d''\circ d'-d'\circ d''\circ d-d''\circ d'\circ d,d\circ d'\circ D''-d\circ D''\circ d'-d'\circ D''\circ d-D''\circ d'\circ d)\end{aligned}$$

and

$$\begin{aligned}&\left[\left[(d,D),(d',D')\right],(d'',D'')\right]+\left[(d',D'),\left[(d,D),(d'',D'')\right]\right]\\&=\left[(d\circ d'-d'\circ d,d\circ D'-D'\circ d),(d'',D'')\right]+\left[(d',D'),(d\circ d''-d''\circ d,d\circ D''-D''\circ d)\right]\\&=((d\circ d'-d'\circ d)\circ d''-d''\circ(d\circ d'-d'\circ d),(d\circ d'-d'\circ d)\circ D''-D''\circ(d\circ d'-d'\circ d))\\&\quad+(d'\circ(d\circ d''-d''\circ d)-(d\circ d''-d''\circ d)\circ d',d'\circ(d\circ D''-D''\circ d)-(d\circ D''-D''\circ d)\circ d')\\&=(d\circ d'\circ d''-d\circ d''\circ d'-d'\circ d''\circ d-d''\circ d'\circ d,d\circ d'\circ D''-d\circ D''\circ d'-d'\circ D''\circ d-D''\circ d'\circ d).\end{aligned}$$

Thus,

$$\left[(d,D),\left[(d',D'),(d'',D'')\right]\right]=\left[\left[(d,D),(d',D')\right],(d'',D'')\right]+\left[(d',D'),\left[(d,D),(d'',D'')\right]\right]$$

which implies that the Leibniz identity holds. Therefore, Bider(L) is a left Leibniz algebra.

An algorithm for finding derivations

Motivated by the program in [10], we consider the n-dimensional Leibniz algebra L with the basis $\{e_1, e_2, \ldots, e_n\}$ with $\left[e_i,e_j\right]=\sum_{k=1}^{n} L_{ij}^k e_k$ for $1\le i,j\le n$. Let d be a derivation and $d(e_i)=\sum_{k=1}^{n} d_{ki}e_k$ for $1\le i\le n$. We construct the system of equations on the basis $\{e_1, e_2, \ldots, e_n\}$ as follows:

$$d(\left[e_i,e_j\right])=\sum_{m=1}^{n}\left(\sum_{k=1}^{n} L_{ij}^k d_{mk}\right)e_m,$$

$$\left[d(e_i),e_j\right]+\left[e_i,d(e_j)\right]=\sum_{m=1}^{n}\left(\sum_{k=1}^{n}\left(L_{kj}^m d_{ki}+L_{ik}^m d_{kj}\right)\right)e_m,$$

for all $1\le i,j\le n$. Due to the property of derivation, we have

$$\sum_{k=1}^{n} L_{ij}^k d_{mk}=\sum_{k=1}^{n}\left(L_{kj}^m d_{ki}+L_{ik}^m d_{kj}\right)$$

for all $1 \leq m \leq n$. Hence, we obtain the following algorithm to find derivations of a given finite dimensional complex Leibniz algebra using Maple. We will implement the algorithm in Maple to find derivations of two and three-dimensional non-Lie Leibniz algebras in the last section.

```
derivation := proc(L, n)
local i, j, k, t, s1, s2, m, d, sols, eqns, Der;
eqns := {};
d := matrix(n, n);
Der := matrix(n, n);
for i to n do
    for j to n do
        for m to n do
            s1 := sum(L[i, j, k]*d[m, k], k = 1 .. n);
            s2 := sum(L[k, j, m]*d[k, i]+L[i, k, m]*d[k, j], k = 1 .. n);
            eqns := union(eqns, {s1 = s2});
        end do
    end do
end do;
sols := [solve(eqns, useassumptions)];
t := nops(sols);
for i to t do
    for j to n do
        for k to n do
            Der[k, j] := subs(sols[i], d[k, j]);
        end do
    end do
end do;
print("Derivation := ", Der);
end proc:
```

Remark 1 In [10], Said Husain, Rakhimov and Basri also provided an algorithm for finding derivations of Leibniz algebras. However, it seems there is a typo in their program causing an error on Maple.

Example 4 Let L be a two-dimensional Leibniz algebra with the non-zero bracket in L given by $[e_1, e_1] = e_2$. By using the above algorithm in Maple 15, we obtain derivations of L as follows:

Input:

```
L := array(sparse, 1..2, 1..2, 1..2, [(1,1,2)=1]):
derivation(L,2);
```

Output:

$$\text{Derivation} := \begin{pmatrix} d_{11} & 0 \\ d_{21} & 2d_{11} \end{pmatrix}$$

An algorithm for finding antiderivations

Consider the n -dimensional Leibniz algebra L with the basis $\{e_1, e_2, \ldots, e_n\}$ with $[e_i, e_j] = \sum_{k=1}^{n} L_{ij}^k e_k$ for $1 \leq i, j \leq n$. Let D be an antiderivation and $D(e_i) = \sum_{k=1}^{n} D_{ki} e_k$ for $1 \leq i \leq n$. We construct the system of equations on the basis $\{e_1, e_2, \ldots, e_n\}$ as follows:

$$D([e_i, e_j]) = \sum_{m=1}^{n} \left(\sum_{k=1}^{n} L_{ij}^k D_{mk} \right) e_m,$$

$$[e_i, D(e_i)] - [e_j, d(e_i)] = \sum_{m=1}^{n} \left(\sum_{k=1}^{n} L_{ik}^m D_{kj} - \sum_{k=1}^{n} L_{jk}^m D_{ki} \right) e_m,$$

for all $1 \leq i, j \leq n$. Due to the property of antiderivation, we have

$$\sum_{k=1}^{n} L_{ij}^k D_{mk} = \sum_{k=1}^{n} L_{ik}^m D_{kj} - \sum_{k=1}^{n} L_{jk}^m D_{ki}$$

for all $1 \leq m \leq n$. Hence, we obtain the following algorithm to find antiderivations of a given finite dimensional complex Leibniz algebra using Maple. We will implement the algorithm in Maple to find antiderivations of two and three-dimensional non-Lie Leibniz algebras in the last section.

```
antiderivation := proc(L, n)
local i, j, k, t, s1, s2, m, D, sols, eqns, AntiDer;
eqns := {};
D := matrix(n, n);
AntiDer := matrix(n, n);
for i to n do
    for j to n do
        for m to n do
            s1 := sum(L[i, j, k]*D[m, k], k = 1 .. n);
            s2 := sum(L[i, k, m]*D[k, j], k = 1 .. n)
                  -sum(L[j, k, m]*D[k, i], k = 1 .. n);
            eqns := union(eqns, {s1 = s2});
        end do
    end do
end do;
sols := [solve(eqns, useassumptions)];
t := nops(sols);
for i to t do
    for j to n do
        for k to n do
            AntiDer[k, j] := subs(sols[i], D[k, j]);
        end do
    end do
end do;
print("AntiDerivation := ", AntiDer);
end proc:
```

Example 5 Let L be a two-dimensional Leibniz algebra with the non-zero bracket in L given by $\left[e_1, e_1\right] = e_2$. By using the above algorithm in Maple 15, we obtain antiderivations of L as follows:

```
Input:

L := array(sparse, 1..2, 1..2, 1..2, [(1,1,2)=1]):
antiderivation(L,2);

Output:
```

$$\text{AntiDerivation} := \begin{pmatrix} D_{11} & 0 \\ D_{21} & 0 \end{pmatrix}$$

Derivations, antiderivations and biderivations of low-dimensional non-Lie Leibniz algebras

In this section, we apply the algorithms from previous sections to low-dimensional non-Lie complex Leibniz algebras. We first observe that if L is a non-Lie Leibniz algebra, then Leib(L) $\neq \{0\}$. Also, Leib(L) $\neq L$ because Leib(L) is abelian. Thus, there does not exist any non-Lie Leibniz algebra with dim(L) $=1$. Let L be a non-Lie Leibniz algebra of dimension 2. Let B denote the ordered basis for L given by $B=\{e_1,e_2\}$. By [7], L is isomorphic to one of the following algebras with the nonzero brackets given:

$$L_2^1: [e_1,e_1]=e_2, \quad L_2^2: [e_1,e_1]=[e_1,e_2]=e_2.$$

Applying the algorithms, we obtain the following result.

The derivations and antiderivations of two-dimensional non-Lie Leibniz algebras are given as follows.

Table 1 Derivations and antiderivations of two-dimensional non-Lie Leibniz algebras.

L	Der(L)	dim Der(L)	Antider(L)	dim Antider(L)
L_2^1	$\begin{pmatrix} d_{11} & 0 \\ d_{21} & 2d_{11} \end{pmatrix}$	2	$\begin{pmatrix} D_{11} & 0 \\ D_{21} & 0 \end{pmatrix}$	2
L_2^2	$\begin{pmatrix} 0 & 0 \\ d_{22} & d_{22} \end{pmatrix}$	1	$\begin{pmatrix} D_{11} & 0 \\ D_{21} & 0 \end{pmatrix}$	2

To find a basis for Bider(L) , we let $d\in$ Der(L) and $D\in$ Antider(L) . Thus $d=\alpha_1 d_1+\alpha_2 d_2$ and $D=\beta_1 D_1+\beta_2 D_2$ where $\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbb{F}$ satisfying the following conditions $[d(e_i),e_j]=[D(e_i),e_j]$ for $1\le i,j\le 2$. By straightforward computations, we obtain the following result.

The biderivations of two-dimensional non-Lie Leibniz algebras are given as follows.

Table 2 Biderivations of two-dimensional non-Lie Leibniz algebras.

L	Bider(L)	dim Bider(L)
L_2^1	$\left(\begin{pmatrix} d_{11} & 0 \\ d_{21} & 2d_{11} \end{pmatrix}, \begin{pmatrix} d_{11} & 0 \\ D_{21} & 0 \end{pmatrix}\right)$	3
L_2^2	$\left(\begin{pmatrix} 0 & 0 \\ d_{22} & d_{22} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ D_{21} & 0 \end{pmatrix}\right)$	2

Now let L be a non-Lie Leibniz algebra of dimension 3. Let B denote the ordered basis for L given by $B=\{e_1,e_2,e_3\}$. By [7], L is isomorphic to one of the following algebras with the nonzero brackets given:

L_3^1: $[e_1,e_1]=e_2, [e_1,e_2]=e_3,$

L_3^2: $[e_1,e_1]=e_3,$

L_3^3: $[e_1,e_1]=e_3, [e_2,e_2]=e_3,$

L_3^4: $[e_1, e_2] = e_3, [e_2, e_1] = -e_3, [e_2, e_2] = e_3,$
L_3^5: $[e_1, e_2] = e_3, [e_2, e_1] = \alpha e_3,\ \alpha \in \mathbb{F} \setminus \{1, -1\},$
L_3^6: $[e_1, e_3] = e_3,$
L_3^7: $[e_1, e_3] = \alpha e_3,\ \alpha \in \mathbb{F} \setminus \{0\}, [e_1, e_2] = e_2, [e_2, e_1] = -e_2,$
L_3^8: $[e_1, e_2] = e_2, [e_2, e_1] = -e_2, [e_1, e_1] = e_3,$
L_3^9: $[e_1, e_3] = 2e_3, [e_2, e_2] = e_3, [e_1, e_2] = e_2, [e_2, e_1] = -e_2, [e_1, e_1] = e_3,$
L_3^{10}: $[e_1, e_2] = e_2, [e_1, e_3] = \alpha e_3,\ \alpha \in \mathbb{F} \setminus \{0\},$
L_3^{11}: $[e_1, e_3] = e_3 + e_2, [e_1, e_2] = e_2,$
L_3^{12}: $[e_1, e_3] = e_2, [e_1, e_2] = e_2,\ [e_1, e_1] = e_3.$

Applying the algorithms, we obtain the following result.

The derivations and antiderivations of three-dimensional non-Lie Leibniz algebras are given as follows.

Table 3 Derivations and antiderivations of three-dimensional non-Lie Leibniz algebras.

L	Der(L)	dim Der(L)	Antider(L)	dim Antider(L)
L_3^1	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{32} & 2d_{11} & 0 \\ d_{31} & d_{32} & 3d_{11} \end{pmatrix}$	3	$\begin{pmatrix} D_{11} & 0 & 0 \\ D_{21} & 0 & 0 \\ D_{31} & 0 & 0 \end{pmatrix}$	3
L_3^2	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & 0 \\ d_{31} & d_{32} & 2d_{11} \end{pmatrix}$	5	$\begin{pmatrix} D_{11} & 0 & 0 \\ D_{21} & D_{22} & 0 \\ D_{31} & D_{32} & 0 \end{pmatrix}$	5
L_3^3	$\begin{pmatrix} d_{22} & -d_{21} & 0 \\ d_{21} & d_{22} & 0 \\ d_{31} & d_{32} & 2d_{22} \end{pmatrix}$	4	$\begin{pmatrix} D_{11} & D_{21} & 0 \\ D_{21} & D_{22} & 0 \\ D_{31} & D_{32} & 0 \end{pmatrix}$	5
L_3^4	$\begin{pmatrix} d_{22} & d_{12} & 0 \\ 0 & d_{22} & 0 \\ d_{31} & d_{32} & 2d_{22} \end{pmatrix}$	4	$\begin{pmatrix} D_{21} - D_{22} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ D_{31} & D_{32} & 0 \end{pmatrix}$	5
L_3^5	$\begin{pmatrix} d_{33} - d_{22} & 0 & 0 \\ 0 & d_{22} & 0 \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$	4	$\begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{21} & \alpha D_{11} & 0 \\ D_{31} & D_{32} & 0 \end{pmatrix}$	5
L_3^6	$\begin{pmatrix} 0 & 0 & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$	3	$\begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ D_{31} & 0 & 0 \end{pmatrix}$	5
L_3^7	$\begin{pmatrix} 0 & 0 & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$	3	$\begin{pmatrix} 0 & 0 & 0 \\ D_{21} & D_{22} & 0 \\ D_{31} & 0 & 0 \end{pmatrix}$	3

Table 3 Derivations and antiderivations of three-dimensional non-Lie Leibniz algebras. (cont.)

L	Der(L)	dim Der(L)	Antider(L)	dim Antider(L)
L_3^8	$\begin{pmatrix} 0 & 0 & 0 \\ d_{21} & d_{22} & 0 \\ d_{31} & 0 & 0 \end{pmatrix}$	3	$\begin{pmatrix} 0 & 0 & 0 \\ D_{21} & D_{22} & 0 \\ D_{31} & 0 & 0 \end{pmatrix}$	3
L_3^9	$\begin{pmatrix} 0 & 0 & 0 \\ -d_{32} & \frac{d_{33}}{2} & 0 \\ \frac{d_{33}}{2} & d_{32} & d_{33} \end{pmatrix}$	2	$\begin{pmatrix} 0 & 0 & 0 \\ D_{32} & D_{22} & 0 \\ D_{31} & D_{32} & 0 \end{pmatrix}$	3
L_3^{10}	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$, $\alpha \neq 1$	2	$\begin{pmatrix} D_{11} & 0 & 0 \\ D_{21} & 0 & 0 \\ D_{31} & 0 & 0 \end{pmatrix}$	3
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{pmatrix}$, $\alpha = 1$	4		
L_3^{11}	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & d_{33} & d_{23} \\ 0 & 0 & d_{33} \end{pmatrix}$	2	$\begin{pmatrix} D_{11} & 0 & 0 \\ D_{21} & 0 & 0 \\ D_{31} & 0 & 0 \end{pmatrix}$	3
L_3^{12}	$\begin{pmatrix} 0 & 0 & 0 \\ d_{23}-d_{31} & d_{23} & d_{23} \\ d_{31} & 0 & 0 \end{pmatrix}$	2	$\begin{pmatrix} D_{11} & 0 & 0 \\ D_{21} & 0 & 0 \\ D_{31} & 0 & 0 \end{pmatrix}$	3

Remark 2 We observe that by suitable change of basis, the isomorphism classes of derivation algebras in Table 1 and Table 3 coincide with the isomorphism classes given in [8]. For example,

$$\mathrm{Der}\left(L_3^5\right) = \left\{ d_{22}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + d_{31}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + d_{32}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + d_{33}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| d_{ij} \in \mathbb{F} \right\}$$
$$= \left\{ \delta_1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \delta_2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \delta_3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \delta_4\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \middle| \delta_i \in \mathbb{F} \right\}.$$

To find a basis for Bider(L) , we let $d \in$ Der(L) and $D \in$ Antider(L) . Thus $d = \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3$ and $D = \beta_1 D_1 + \beta_2 D_2 + \beta_3 D_3$ where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{F}$ satisfying the following conditions $\left[d\left(e_i\right), e_j\right] = \left[D\left(e_i\right), e_j\right]$ for $1 \leq i, j \leq 3$. We consider each Leibniz algebra isomorphism class case by case. By straightforward computations, we obtain the following result.

The biderivations of three-dimensional non-Lie Leibniz algebras are given as follows.

Table 4 Biderivations of three-dimensional non-Lie Leibniz algebras.

L	Bider(L)	dim Bider(L)
L_3^1	$\left(\begin{pmatrix} d_{11} & 0 & 0 \\ d_{32} & 2d_{11} & 0 \\ d_{31} & d_{32} & 3d_{11} \end{pmatrix}, \begin{pmatrix} d_{11} & 0 & 0 \\ D_{21} & 0 & 0 \\ D_{31} & 0 & 0 \end{pmatrix}\right)$	5
L_3^2	$\left(\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & 0 \\ d_{31} & d_{32} & 2d_{11} \end{pmatrix}, \begin{pmatrix} d_{11} & 0 & 0 \\ D_{21} & D_{22} & 0 \\ D_{31} & D_{32} & 0 \end{pmatrix}\right)$	9
L_3^3	$\left(\begin{pmatrix} d_{22} & 0 & 0 \\ 0 & d_{22} & 0 \\ d_{31} & d_{32} & 2d_{22} \end{pmatrix}, \begin{pmatrix} d_{22} & 0 & 0 \\ 0 & d_{22} & 0 \\ D_{31} & D_{32} & 0 \end{pmatrix}\right)$	5
L_3^4	$\left(\begin{pmatrix} 0 & d_{12} & 0 \\ 0 & 0 & 0 \\ d_{31} & d_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & d_{12} & 0 \\ 0 & 0 & 0 \\ D_{31} & D_{32} & 0 \end{pmatrix}\right)$	5
L_3^5	$\left(\begin{pmatrix} d_{33}-d_{22} & 0 & 0 \\ 0 & d_{22} & 0 \\ d_{31} & d_{32} & d_{33} \end{pmatrix}, \begin{pmatrix} d_{33}-d_{22} & 0 & 0 \\ 0 & d_{22} & 0 \\ D_{31} & D_{32} & 0 \end{pmatrix}\right)$	6
L_3^6	$\left(\begin{pmatrix} 0 & 0 & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ D_{21} & D_{22} & 0 \\ D_{31} & 0 & 0 \end{pmatrix}\right)$	6
L_3^7	$\left(\begin{pmatrix} 0 & 0 & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ d_{21} & d_{22} & 0 \\ D_{31} & 0 & 0 \end{pmatrix}\right)$	4
L_3^8	$\left(\begin{pmatrix} 0 & 0 & 0 \\ d_{21} & d_{22} & 0 \\ d_{31} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ d_{21} & d_{22} & 0 \\ D_{31} & 0 & 0 \end{pmatrix}\right)$	4
L_3^9	$\left(\begin{pmatrix} 0 & 0 & 0 \\ -d_{32} & \frac{d_{33}}{2} & 0 \\ \frac{d_{33}}{2} & d_{32} & d_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -d_{32} & \frac{d_{33}}{2} & 0 \\ D_{31} & -d_{32} & 0 \end{pmatrix}\right)$	3
L_3^{10}	$\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ D_{21} & 0 & 0 \\ D_{31} & 0 & 0 \end{pmatrix}\right), \alpha \neq 1$	4
	$\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ D_{21} & 0 & 0 \\ D_{31} & 0 & 0 \end{pmatrix}\right), \alpha = 1$	6
L_3^{11}	$\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & d_{33} & d_{23} \\ 0 & 0 & d_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ D_{21} & 0 & 0 \\ D_{31} & 0 & 0 \end{pmatrix}\right)$	4
L_3^{12}	$\left(\begin{pmatrix} 0 & 0 & 0 \\ d_{23}-d_{31} & d_{23} & d_{23} \\ d_{31} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ D_{21} & 0 & 0 \\ D_{31} & 0 & 0 \end{pmatrix}\right)$	4

Remark 3 Our Maple software algorithms can be used for any finite-dimensional Leibniz algebra. In this work, we focus on low-dimensional Leibniz algebras due to the computational feasibility of explicit classification and their ability to reveal important structural properties that can guide studies in higher dimensions.

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