Research Article

A Modified Taylor Series Expansion Method for the Second Order Linear Volterra Integro-Differential Equation

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ABSTRACT

In this paper, we used a modified Taylor series expansion method for approximating the solutions of linear second order Volterra Integro-Differential Equation (VIDE). This method transforms the equation to linear system equations that can be solved easily with computer programing. Finally, we showed the efficiency of this method with numerical examples by comparing the approximate solutions with exact solutions.

Keywords: Taylor Series Expansion, Linear Volterra Integro-Differential Equation, Numerical Method, Approximate Solution.

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Introduction

Mathematical modeling of real-life problems usually results in functional equations, e.g., differential equations, integral and integro-differential equations, stochastic equations and others. Integral equation and integro-differential equation are used in applied mathematics or mathematical physics. Integro-differential equation is a hybrid of integral and differential equations which has found extensive applications in sciences and engineering. In particular, the equation arises in fluid dynamics, biological models, and chemical kinetics. Solutions of some integral and integro-differential equations cannot be found, thus numerical methods are required. Many researchers have used various direct or numerical methods to find the approximate solution such as decomposition method, series expansion method, Laplace transform, Tau method and Padé approximation [1-3]. As a result, the numerical solutions sometimes are very complicated to obtain. There is an alternative method of approximating the solution was proposed. One of the alternative approaches is a Taylor series expansion [4-6]. The Taylor series expansion is widely used to approximate the solution of differential equations and integral equations because it is easy, convenient, and fast calculations.

The Volterra integro-Differential Equation (VIDE) was given in form

$$x^{(n)}(s)-\int_0^s k(s,t)x(t)\mathrm{d}t = f(s), \qquad 0\le s\le T\,, \tag{1}$$

with the initial condition $x(0), x'(0),..., x^{(n-1)}(0)$. The VIDE appeared in many physical applications such as glass forming process, nanohydrodynamics, heat transfer, diffusion process and biology species coexisting together [2].

In 2021, Navarasuchitr et al. [7] presented the Taylor series expansion for solving the first order linear VIDE in form

$$x'(s)-\int_0^s k(s,t)x(t)\mathrm{d}t = f(s), \quad x(0)=x_0, \quad 0\le s\le T\,, \tag{2}$$

where the functions $f(s)$ and the kernel $k(s,t)$ are known.
Their technique used the Taylor series expansion to expand $x(t)$ defined as follows,

$$x(t)\approx x(s)+x'(s)(t-s)+...+\frac{x^{(n)}(s)(t-s)^n}{n!}\,. \tag{3}$$

They used equation (2) to approximate $x(s)$ and found that their technique was easy and yielded the accurate solution in a few terms and they also used the same technique for solving Fredholm Integro-Differential Equations (FIDEs) [8] given in form

$$x'(s)-\int_a^b k(s,t)x(t)\mathrm{d}t = f(s), \quad x(0)=x_0\,, \tag{4}$$

where the functions $f(s)$, the kernel $k(s,t)$ are known and a,b are constants. The results of approximation are still accurate in a few terms.

In this paper, we applied Taylor series expansion based on [7, 8] in the new approach for solving the second order linear VIDEs in form

$$x''(s)-\int_0^s k(s,t)\mathrm{d}t = f(s), \quad x(0)=x_0, x'(0)=x_1, \quad 0\le s\le 1, \quad , \tag{5}$$

where the function $f(s)$ and the kernel $k(s,t)$ are known and $x(s)$ was determined a solution. Also, throughout the paper, we assume that $k \in C^n([0,1]\times[0,1])$, $f \in C^n[0,1]$.

Materials and methods

Leibnitz Rule for Differential of Integrals [2]

Theorem 1 Let $f(x,t)$ be continuous function, and $\dfrac{\partial f}{\partial t}$ be continuous in the domain of xt plane that includes the rectangle $a \le x \le b,\ t_0 \le t \le t_1$, and let

$$F(x) = \int_{g(x)}^{h(x)} f(x,t)\mathrm{d}t\,, \tag{6}$$

then the differential of the integral in (6) exists and is given by

$$F'(x) = \frac{\mathrm{d}F}{\mathrm{d}x} = f(x,h(x))\frac{\mathrm{d}h(x)}{\mathrm{d}x} - f(x,g(x))\frac{\mathrm{d}g(x)}{\mathrm{d}x} + \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x}\mathrm{d}t. \tag{7}$$

Remark 1

1. If $g(x) = a$ and $h(x) = b$ where a and b are constants, then the Leibnitz rule (7) reduces to

$$F'(x) = \frac{\mathrm{d}F}{\mathrm{d}x} = \int_a^b \frac{\partial f(x,t)}{\partial x}\mathrm{d}t\,, \tag{8}$$

   which means that the differentiation and integration can be interchanged.

2. In this paper, we will focus on differentiation of integral in form

$$F(s) = \int_0^s k(s,t)x(t)\mathrm{d}t\,. \tag{9}$$

   In this case, we apply theorem 1 yields

$$F'(s) = k(s,s)x(s) + \int_0^s \frac{\partial k(s,t)}{\partial s} x(t)\mathrm{d}t\,. \tag{10}$$

Reducing Multiple Integrals to Single Integral [2]

Theorem 2 If F is continuous then

$$\int_0^x \int_0^{x_1} F(t)\mathrm{d}t\mathrm{d}x_1 = \int_0^x (x-t)F(t)\mathrm{d}t\,, \tag{11}$$

Corollary 1 The general formula that reduces the multiple integrals to a single integral is given by

$$\int_0^x \int_0^{x_1} \cdots \int_0^{x_{n-1}} u(x_n)\mathrm{d}x_n\mathrm{d}x_{n-1}\cdots\mathrm{d}x_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}u(t)\mathrm{d}t. \tag{12}$$

Numerical Method

We consider the second order linear VIDE in form

$$x''(s) - \int_0^s k(s,t)\mathrm{d}t = f(s), \quad x(0) = x_0, x'(0) = x_1, \quad 0 \le s \le 1, \tag{13}$$

where the functions $f(s)$ and the kernel $k(s,t)$ are known.

We start our method with integrating equation (13) with respect to s, we obtain

$$x'(s) - \int_0^s \int_0^{s_1} k(s_1,t)\mathrm{d}t\mathrm{d}s_1 = \int_0^s f(s_1)\mathrm{d}s_1 + x'(0), \tag{14}$$

and by integrating equation (14) with respect to s, we derive

$$x(s)-\int_0^s\int_0^{s_1}\int_0^{s_2}k(s_2,t)x(t)\mathrm{d}t\mathrm{d}s_2\mathrm{d}s_1=\int_0^s\int_0^{s_1}f(s_2)\mathrm{d}s_2\mathrm{d}s_1+\int_0^s x'(0)\mathrm{d}s+x(0). \tag{15}$$

Using the corollary 1 with equation (14) , we obtain

$$x'(s)-\frac{1}{2!}\int_0^s k^2(s,t)x(t)\mathrm{d}t=\int_0^s f(s_1)\mathrm{d}s_1+x'(0). \tag{16}$$

Like equation (16), the equation (15) transforms to

$$x(s)-\frac{1}{3!}\int_0^s k^3(s,t)x(t)\mathrm{d}t=\int_0^s\int_0^{s_1}f(s_2)\mathrm{d}s_2\mathrm{d}s_1+\int_0^s x'(0)\mathrm{d}s+x(0). \tag{17}$$

Applying Leibnitz rule for differential integrals to equation (13), and differentiating equation (13) n−1 times to obtain

$$x'''(s)-\left[k(s,s)x(s)+\int_0^s k_s'(s,t)x(t)\mathrm{d}t\right]=f'(s). \tag{18}$$

Differentiating equation (18) to derive

$$x^{(4)}(s)-\left[k(s,s)x'(s)+2k_s'(s,s)x(s)+\int_0^s k_s''(s,t)x(t)\mathrm{d}t\right]=f''(s). \tag{19}$$

Again, to equation (19), we have

$$x^{(5)}(s)-\left[k(s,s)x''(s)+3k_s'(s,s)x'(s)+3k_s''(s,s)x(s)+\int_0^s k_s'''(s,t)x(t)\mathrm{d}t\right]=f'''(s) \tag{20}$$

We continue processing the same method and obtain

$$x^{(n)}(s)-\left[k(s,s)x^{(n-3)}(s)+\binom{n-2}{1}k_s'(s,s)x^{(n-4)}(s)+\binom{n-2}{2}k_s''(s,s)x^{(n-5)}(s)+\right.$$
$$\left.\ldots+\binom{n-2}{n-3}k_s^{(n-3)}(s,s)x(s)+\int_0^s k_s^{(n-2)}(s,t)x(t)\mathrm{d}t\right]=f^{(n-2)}(s) \tag{21}$$

where $k_s^{(i)}(s,t)=\dfrac{\partial^i k(s,t)}{\partial s^i}, i=1,2,\ldots,n$.

By substituting equation (3) for each $x(t)$ in equation (13), (16) – (21) respectively, we have

$$x(s)-\frac{1}{3!}\int_0^s k^3(s,t)\left[x(s)+x'(s)(t-s)+\ldots+\frac{1}{n!}x^{(n)}(s)(t-s)^n\right]\mathrm{d}t=\int_0^s\int_0^{s_1}f(s_2)\mathrm{d}s_2\mathrm{d}s_1+\int_0^s x'(0)\mathrm{d}s+x(0)$$

$$x'(s)-\frac{1}{2!}\int_0^s k^2(s,t)\left[x(s)+x'(s)(t-s)+\ldots+\frac{1}{n!}x^{(n)}(s)(t-s)^n\right]\mathrm{d}t=\int_0^s f(s_1)\mathrm{d}s_1+x'(0)$$

$$x''(s)-\int_0^s k(s,t)\left[x(s)+x'(s)(t-s)+\ldots+\frac{1}{n!}x^{(n)}(s)(t-s)^n\right]\mathrm{d}t=f(s)$$

$$x'''(s)-k(s,s)x(s)-\int_0^s k'_s(s,t)\left[x(s)+x'(s)(t-s)+\ldots+\frac{1}{n!}x^{(n)}(s)(t-s)^n\right]\mathrm{d}t=f'(s)$$

$$x^{(4)}(s)-k(s,s)x'(s)-2k'_s(s,s)x(s)-\int_0^s k''_s(s,t)\left[x(s)+x'(s)(t-s)+\ldots+\frac{1}{n!}x^{(n)}(s)(t-s)^n\right]\mathrm{d}t=f''(s)$$

$$x^{(5)}(s)-k(s,s)x''(s)-3k'_s(s,s)x(s)-3k''_s(s,s)x(s)-\int_0^s k'''_s(s,t)\left[x(s)+x'(s)(t-s)+\ldots+\frac{1}{n!}x^{(n)}(s)(t-s)^n\right]\mathrm{d}t=f'''(s)$$

$$\vdots$$

$$x^{(n)}(s)-k(s,s)x^{(n-3)}(s)-\binom{n-2}{1}k'_s(s,s)x^{(n-4)}(s)-\binom{n-2}{2}k_s''(s,s)x^{(n-5)}(s)-$$
$$\ldots-\binom{n-2}{n-3}k_s^{(n-3)}(s,s)x(s)-\int_0^s k_s^{(n-2)}(s,t)\left[x(s)+x'(s)(t-s)+\ldots+\frac{1}{n!}x^{(n)}(s)(t-s)^n\right]\mathrm{d}t=f^{(n-2)}(s)$$

To solve for $x(s),x'(s),\ldots,x^{(n)}(s)$, the system from above can be written in the following matrix form

$$\begin{pmatrix}
1-\frac{1}{3!}\int_0^s k^3(s,t)\mathrm{d}t & -\frac{1}{3!}\int_0^s k^3(s,t)(t-s)\mathrm{d}t & \cdots & \frac{1}{3!}\frac{1}{n!}\int_0^s k^3(s,t)(t-s)^n\mathrm{d}t \\
-\frac{1}{2!}\int_0^s k^2(s,t)\mathrm{d}t & 1-\frac{1}{2!}\int_0^s k^2(s,t)(t-s)\mathrm{d}t & \cdots & \frac{1}{2!}\frac{1}{n!}\int_0^s k^2(s,t)(t-s)^n\mathrm{d}t \\
-\int_0^s k(s,t)\mathrm{d}t & -\int_0^s k(s,t)(t-s)\mathrm{d}t & \cdots & -\frac{1}{n!}\int_0^s k(s,t)(t-s)^n\mathrm{d}t \\
-k(s,s)-\int_0^s k_s'(s,t)\mathrm{d}t & -\int_0^s k_s'(s,t)(t-s)\mathrm{d}t & \cdots & -\frac{1}{n!}\int_0^s k_s'(s,t)(t-s)^n\mathrm{d}t \\
-2k_s'(s,s)-\int_0^s k_s''(s,t)\mathrm{d}t & -k(s,s)-\int_0^s k_s''(s,t)(t-s)\mathrm{d}t & \cdots & -\frac{1}{n!}\int_0^s k_s''(s,t)(t-s)^n\mathrm{d}t \\
-3k_s'''(s,s)-\int_0^s k_s'''(s,t)\mathrm{d}t & -3k_s'(s,s)-\int_0^s k_s'''(s,t)(t-s)\mathrm{d}t & \cdots & -\frac{1}{n!}\int_0^s k_s'''(s,t)(t-s)^n\mathrm{d}t \\
\vdots & \vdots & \ddots & \vdots \\
-\binom{n-2}{n-3}k_s^{(n-3)}(s,s)- & -\binom{n-2}{n-4}k_s^{(n-4)}(s,s)- & \cdots & 1-\frac{1}{n!}\int_0^s k_s^{(n-2)}(s,t)(t-s)^n\mathrm{d}t \\
\int_0^s k_s^{(n-2)}(s,t)\mathrm{d}t & \int_0^s k_s^{(n-2)}(s,t)(t-s)\mathrm{d}t & &
\end{pmatrix}$$

$$\times\begin{pmatrix} x(s) \\ x'(s) \\ x''(s) \\ x'''(s) \\ x^{(4)}(s) \\ x^{(5)}(s) \\ \vdots \\ x^{(n)}(s) \end{pmatrix}=\begin{pmatrix} \int_0^s\int_0^{s_1} f(s_2)\mathrm{d}s_2\mathrm{d}s_1+\int_0^s x'(0)\mathrm{d}s+x(0) \\ \int_0^2 f(s_1)\mathrm{d}s_1+x'(0) \\ f(s) \\ f'(s) \\ f''(s) \\ f'''(s) \\ \vdots \\ f^{(n-2)}(s) \end{pmatrix}. \tag{22}$$

We use the reduced row echelon form to find the solutions of $x(s),x'(s),\ldots,x^{(n)}(s)$.

Results and Discussion

Numerical Results

Example 1: Consider the VIDE

$$x''(s)-\int_0^s (s-t)x(t)\mathrm{d}t = 1+s, \quad x(0)=x'(0)=1$$

which has the exact solution $x(s)=\mathrm{e}^s$.

We use Taylor series order n = 1, 2 and 3 and apply with the system of equation (22) for solving $x(s)$.

With n = 1.

The system of equation is

$$\begin{pmatrix} 1-\frac{1}{24}s^4 & \frac{1}{30}s^5 \\ -\frac{1}{6}s^3 & 1+\frac{1}{8}s^4 \end{pmatrix}\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}s^2+\frac{1}{6}s^3+s+1 \\ s+\frac{1}{2}s^2+1 \end{pmatrix}.$$

With n = 2.

The system of equation is

$$\begin{pmatrix} 1-\frac{1}{24}s^4 & \frac{1}{30}s^5 & -\frac{1}{72}s^6 \\ -\frac{1}{6}s^3 & 1+\frac{1}{8}s^4 & -\frac{1}{20}s^5 \\ -\frac{1}{2}s^2 & \frac{1}{3}s^3 & 1-\frac{1}{8}s^4 \end{pmatrix}\begin{pmatrix} x(s) \\ x'(s) \\ x''(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}s^2+\frac{1}{6}s^3+s+1 \\ s+\frac{1}{2}s^2+1 \\ s+1 \end{pmatrix}.$$

With n = 3.

The system of equation is

$$\begin{pmatrix} 1-\frac{1}{24}s^4 & \frac{1}{30}s^5 & -\frac{1}{72}s^6 & \frac{1}{252}s^7 \\ -\frac{1}{6}s^3 & 1+\frac{1}{8}s^4 & -\frac{1}{20}s^5 & \frac{1}{72}s^6 \\ -\frac{1}{2} & \frac{1}{3}s^3 & 1-\frac{1}{8}s^4 & \frac{1}{3}s^5 \\ -s & \frac{1}{2}s^2 & -\frac{1}{6}s^3 & 1+\frac{1}{24}s^4 \end{pmatrix}\begin{pmatrix} x(s) \\ x'(s) \\ x''(s) \\ x'''(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}s^2+\frac{1}{6}s^3+s+1 \\ s+\frac{1}{2}s^2+1 \\ s+1 \\ 1 \end{pmatrix}.$$

The numerical errors between our approximation and exact solution by using Taylor series with n = 1, 2 and 3 are shown in Table 1. In Figure 1 also shows the exact and approximate solution n = 1, 2 and 3.

Table 1 The absolute and relative errors compare with the exact solution for example 1.

	Absolute Errors			Relative Errors		
s	$n = 1$	$n = 2$	$n = 3$	$n = 1$	$n = 2$	$n = 3$
0	0	0	0	0	0	0
0.1	$1.5\text{x}10^{-8}$	$1\text{x}10^{-9}$	0	$1.36\text{x}10^{-8}$	$9\text{x}10^{-10}$	0
0.2	$1.02\text{x}10^{-6}$	$6\text{x}10^{-8}$	$2\text{x}10^{-9}$	$8.35\text{x}10^{-7}$	$4.91\text{x}10^{-8}$	$1.6\text{x}10^{-9}$
0.3	0.00001256	$1.09\text{x}10^{-6}$	$7.4\text{x}10^{-8}$	$9.3\text{x}10^{-6}$	$8.07\text{x}10^{-7}$	$5.48\text{x}10^{-8}$
0.4	0.00007580	$8.91\text{x}10^{-6}$	$7.92\text{x}10^{-7}$	0.00005081	$5.97\text{x}10^{-6}$	$5.3\text{x}10^{-7}$
0.5	0.00031028	0.00004610	$5.19\text{x}10^{-6}$	0.00018819	0.00002796	$3.14\text{x}10^{-6}$
0.6	0.00099214	0.00017942	0.00002396	0.00054449	0.00009846	0.00004441
0.7	0.00267119	0.00057448	0.00008944	0.00132647	0.00285278	0.00012717
0.8	0.00632896	0.00159675	0.00028304	0.00284378	0.00071746	0.00025611
0.9	0.01357147	0.00398970	0.00078979	0.00551774	0.00162209	0.00032110
1.0	0.02683677	0.00918255	0.00199546	0.00987269	0.003378071	0.00073408

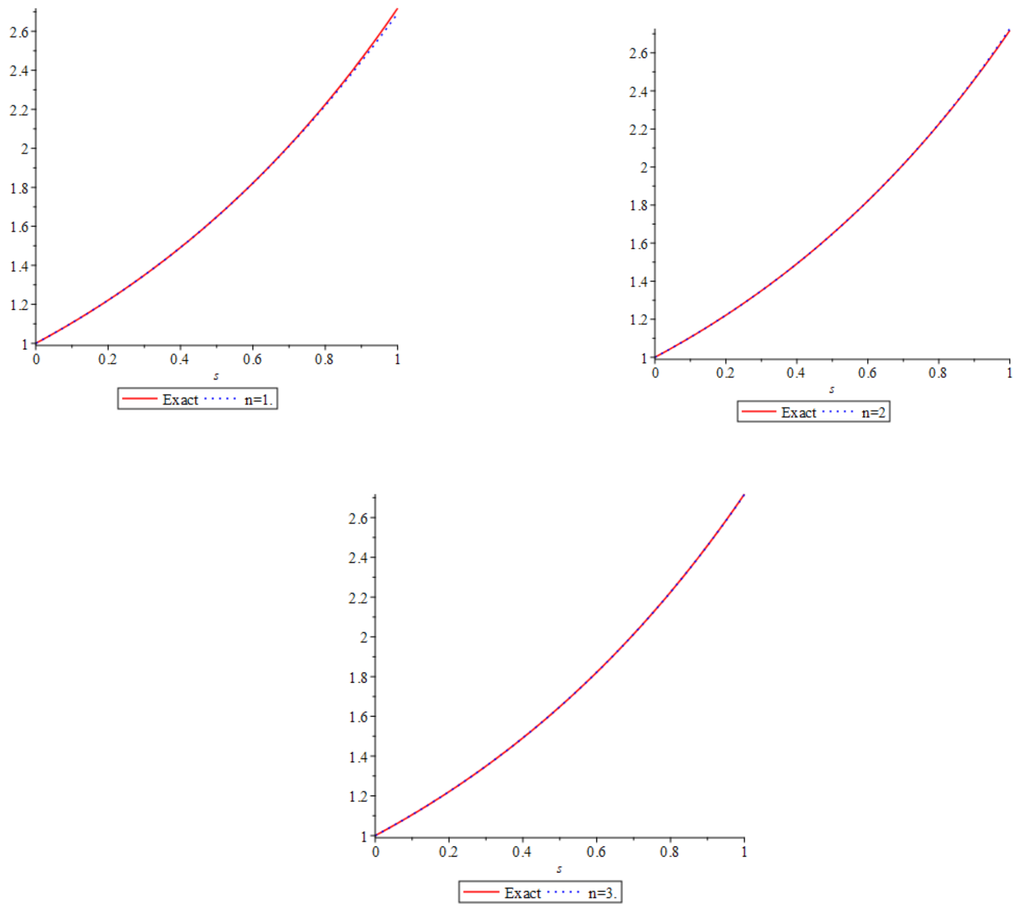

Figure 1 Results for example 1 with n = 1, 2 and 3.

The results of expansion found that n = 3 is the best for the approximation.

Example 2: Consider the VIDE

$$x''(s)-\int_0^s (s-t)x(t)\mathrm{dt} \;=\; 2-s-\frac{s^4}{12}, \quad x(0)=x'(0)=1$$

which has the exact solution $x(s)=s^2+\sin(s)$.

We use Taylor series order n = 1, 2 and 3 and apply with the system of equation (22) for solving $x(s)$.

With n = 1.

The system of equation is

$$\begin{pmatrix} 1-\frac{1}{24}s^4 & \frac{1}{30}s^5 \\ -\frac{1}{6}s^3 & 1+\frac{1}{8}s^4 \end{pmatrix}\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}=\begin{pmatrix} s^2-\frac{1}{6}s^3-\frac{1}{360}s^6+s \\ 2s-\frac{1}{2}s^2-\frac{1}{60}s^5+1 \end{pmatrix}.$$

With n = 2.

The system of equation is

$$\begin{pmatrix} 1-\frac{1}{24}s^4 & \frac{1}{30}s^5 & -\frac{1}{72}s^6 \\ -\frac{1}{6}s^3 & 1+\frac{1}{8}s^4 & -\frac{1}{20}s^5 \\ -\frac{1}{2}s^2 & \frac{1}{3}s^3 & 1-\frac{1}{8}s^4 \end{pmatrix}\begin{pmatrix} x(s) \\ x'(s) \\ x''(s) \end{pmatrix}=\begin{pmatrix} s^2-\frac{1}{6}s^3-\frac{1}{360}s^6+s \\ 2s-\frac{1}{2}s^2-\frac{1}{60}s^5+1 \\ 2-s-\frac{1}{12}s^4 \end{pmatrix}.$$

With n = 3.

The system of equation is

$$\begin{pmatrix} 1-\frac{1}{24}s^4 & \frac{1}{30}s^5 & -\frac{1}{72}s^6 & \frac{1}{252}s^7 \\ -\frac{1}{6}s^3 & 1+\frac{1}{8}s^4 & -\frac{1}{20}s^5 & \frac{1}{72}s^6 \\ -\frac{1}{2}s^2 & \frac{1}{3}s^3 & 1-\frac{1}{8}s^4 & \frac{1}{30}s^5 \\ -s & \frac{1}{2}s^2 & -\frac{1}{6}s^3 & 1+\frac{1}{24}s^4 \end{pmatrix}\begin{pmatrix} x(s) \\ x'(s) \\ x''(s) \\ x'''(s) \end{pmatrix}=\begin{pmatrix} s^2-\frac{1}{6}s^3-\frac{1}{360}s^6+s \\ 2s-\frac{1}{2}s^2-\frac{1}{60}s^5+1 \\ 2-s-\frac{1}{12}s^4 \\ -1-\frac{1}{3}s^3 \end{pmatrix}.$$

The numerical errors between our approximation and exact solution by using Taylor series with n = 1, 2 and 3 are shown in Table 2. In Figure 2 also shows the exact and approximate solution n = 1, 2 and 3.

Table 2 The absolute and relative errors compare with the exact solution for example 2.

	Absolute Errors			Relative Errors		
s	n = 1	n = 2	n = 3	n = 1	n = 2	n = 3
0	0	0	0	0	0	0
0.1	$2.67\text{x}10^{-8}$	$3\text{x}10^{-10}$	0	$2.43\text{x}10^{-7}$	$2.7\text{x}10^{-9}$	0
0.2	$1.65\text{x}10^{-6}$	$5.01\text{x}10^{-8}$	$3\text{x}10^{-10}$	$6.91\text{x}10^{-6}$	$2.09\text{x}10^{-7}$	$1.3\text{x}10^{-9}$
0.3	0.00001808	$8.431\text{x}10^{-7}$	$1.39\text{x}10^{-8}$	0.00004689	0.0000021	$3.61\text{x}10^{-8}$
0.4	0.00009760	$6.1784\text{x}10^{-6}$	$1.833\text{x}10^{-7}$	0.00177642	0.0000112	$3.33\text{x}10^{-7}$
0.5	0.00035680	0.000028633	$1.3515\text{x}10^{-6}$	0.00048915	0.0000392	0.00000185
0.6	0.00101838	0.000099095	$6.8795\text{x}10^{-6}$	0.00110137	0.0001071	0.00000742
0.7	0.00244664	0.000279842	0.000027103	0.00215711	0.0002467	0.00002389
0.8	0.00517325	0.000679798	0.000088461	0.00381126	0.0005008	0.00006517
0.9	0.00990470	0.001469639	0.000249881	0.00621636	0.0009223	0.00015682
1.0	0.01750430	0.002893361	0.000629393	0.00950560	0.0015712	0.00108362

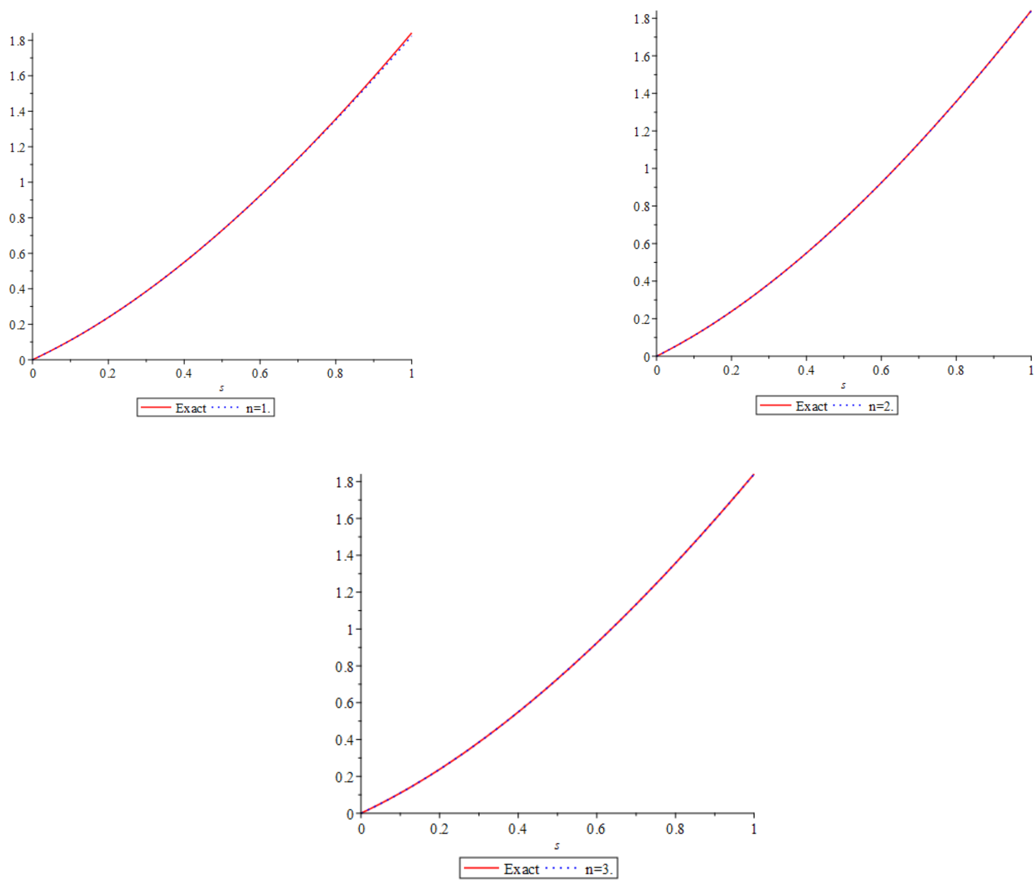

Figure 2 Results for example 2 with n = 1, 2 and 3.

The results of expansion found that n = 3 is the best for the approximation.

Example 3: Consider the VIDE

$$x''(s)-\int_0^s (s-t)x(t)\mathrm{d}t \;=\; -1-s, \quad x(0)=x'(0)=1$$

which has the exact solution $x(s)=\cos(s)+\sin(s)$.

We use Taylor series order n = 1 and 2 and apply with the system of equation (22) for solving $x(s)$.

With n = 1.

The system of equation is

$$\begin{pmatrix} 1-\frac{1}{24}s^4 & \frac{1}{30}s^5 \\ -\frac{1}{6}s^3 & 1+\frac{1}{8}s^4 \end{pmatrix}\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}=\begin{pmatrix} -\frac{1}{2}s^2-\frac{1}{6}s^3+s+1 \\ -s-\frac{1}{2}s^2+1 \end{pmatrix}.$$

With n = 2.

The system of equation is

$$\begin{pmatrix} 1-\frac{1}{24}s^4 & \frac{1}{30}s^5 & -\frac{1}{72}s^6 \\ -\frac{1}{6}s^3 & 1+\frac{1}{8}s^4 & -\frac{1}{20}s^5 \\ -\frac{1}{2}s^2 & \frac{1}{3}s^3 & 1-\frac{1}{8}s^4 \end{pmatrix}\begin{pmatrix} x(s) \\ x'(s) \\ x''(s) \end{pmatrix}=\begin{pmatrix} -\frac{1}{2}s^2-\frac{1}{6}s^3+s+1 \\ -s-\frac{1}{2}s^2+1 \\ -1-s \end{pmatrix}.$$

The numerical errors between our approximation and exact solution by using Taylor series with n = 1 and 2 are shown in Table 3. In Figure 3 also show the exact and approximate solution n = 1 and 2.

Table 3 The absolute errors compare with the exact solution for example 3.

	Absolute Errors		Relative Errors	
s	n = 1	n = 2	n = 1	n = 2
0	0	0	0	0
0.1	$1.5\text{x}10^{-8}$	0	$1.37\text{x}10^{-8}$	0
0.2	$1.005\text{x}10^{-6}$	$4.2\text{x}10^{-8}$	$8.52\text{x}10^{-7}$	$3.56\text{x}10^{-8}$
0.3	0.000012015	$6.41\text{x}10^{-7}$	$9.6\text{x}10^{-6}$	$5.12\text{x}10^{-7}$
0.4	0.000070233	$4.18\text{x}10^{-6}$	0.00005359	$3.18\text{x}10^{-6}$
0.5	0.000276338	0.000016838	0.00020363	0.0000124
0.6	0.000843488	0.000048850	0.00060683	0.0000351
0.7	0.002153528	0.000109318	0.00152834	0.0000775
0.8	0.004808073	0.000188666	0.00340018	0.0001334
0.9	0.009656288	0.000221366	0.00687311	0.0001575
1.0	0.017778135	0.000015225	0.01286617	0.0000110

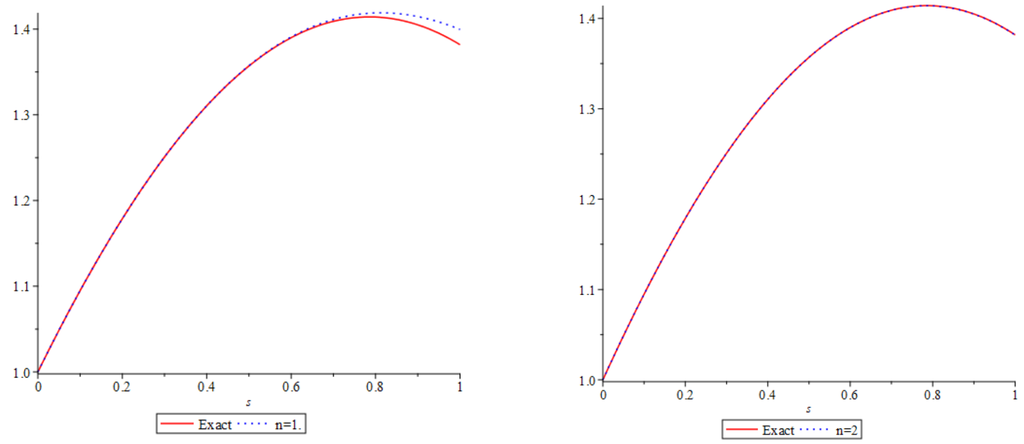

Figure 3 Results for example 3 with $n = 1$ and 2.

The results of expansion found that n = 2 is the best for the approximation.

Conclusions

As illustrated in the examples of this paper, the modified Taylor series method is a powerful procedure for solving the second order linear VIDEs. This method is easy to use and yields an accurate solution in a few terms.

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