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An Intermediate Value Theorem for Graph Parameters

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ABSTRACT

Let $\mathcal{G}$ be the class of all graphs and $\mathcal{J} \subseteq \mathcal{G}$. A graph parameter π is said to satisfy an *intermediate value theorem over a class of graphs* $\mathcal{J}$ if $G, H \in \mathcal{J}$ with $\pi(G) < \pi(H)$, then for every integer k with $\pi(G) \leq k \leq \pi(H)$ there is a graph $K \in \mathcal{J}$ such that $\pi(K) = k$. If a graph parameter π satisfies an intermediate value theorem over $\mathcal{J}$, then we write $(\pi, \mathcal{J}) \in \mathsf{IVT}$. Thus if $(\pi, \mathcal{J}) \in \mathsf{IVT}$, then $\{\pi(G) : G \in \mathcal{J}\}$ is uniquely determined by $\min(\pi, \mathcal{J}) := \min\{\pi(G) : G \in \mathcal{J}\}$ and $\max(\pi, \mathcal{J}) := \max\{\pi(G) : G \in \mathcal{J}\}$. The problem of finding $\min(\pi, \mathcal{J})$ and $\max(\pi, \mathcal{J})$ is called the *extremal problem in graph theory.* We will discuss our results in this direction. Some open problems are also reviewed.

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1. Introduction

Only finite simple graphs are considered in this paper. For the most part, our notation and terminology follows that of Bondy and Murty [1].

Let $\mathcal{J}$ be a class of non-isomorphic graphs. A *graph transformation on* $\mathcal{J}$ is a subset of $\mathcal{J} \times \mathcal{J}$. Let ρ be a graph transformation on $\mathcal{J}$. We can define the *ρ-graph* having $\mathcal{J}$ as its vertex set and there is a directed edge from G to H if and only if $(G, H) \in \rho$. If ρ is symmetric, it yields an undirected graph and otherwise a directed graph.

Harary [2] used a graph transformation called a *fundamental exchange* or an *edge exchange* as follows: Let G be a connected graph of order $n \geq 3$. The *tree graph,* $\mathbf{T}(G)$, of G is defined by specifying $V(\mathbf{T}(G))$ as the set of all spanning trees of G, and two vertices $T_1; T_2 \in V(\mathbf{T}(G))$ are adjacent in $\mathbf{T}(G)$ if and only if T_1 and T_2 have exactly $n - 2$ edges in common. This is an example of an undirected ρ-graph. It was proved by Harary [2] that the tree graph $\mathbf{T}(G)$ is connected.

A non-increasing sequence $\mathbf{d} = (d_1, d_2,...,d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G. In this case, G is called a *realization* of $\mathbf{d}$. A degree sequence of an r-regular graph of order n is denoted by r^n.

Let G be a graph. For the distinct vertices $a, b, c,$ and d in $V(G)$ such that ab and cd are edges in G while ac and bd are not edges in G. Define $G^{\sigma(a,b,c,d)}$, simply written G^{σ}, to be the graph obtained from G by deleting the edges ab and cd and adding the edges ac and bd. The operation $\sigma(a,b,c,d)$ is called a *switching operation.* For a graphic degree sequence $\mathbf{d}$, let $\mathcal{R}(\mathbf{d})$ and $\mathcal{CR}(\mathbf{d})$ be the sets of non-isomorphic realizations and connected realizations of $\mathbf{d}$, respectively. The $\Sigma(\mathbf{d})$ is defined as a relation on $\mathcal{R}(\mathbf{d})$ as $(G, H) \in \Sigma(\mathbf{d})$ if $G \not\cong H$ and there is a switching σ on G such that $H = G^{\sigma}$. Thus the $\Sigma(\mathbf{d})$-graph is simple. The concept of $\Sigma(\mathbf{d})$-graph was introduced and developed in a joint paper by Eggleton and Holton [3]. It provides a structured way to examine all the graphs which “realize” a given degree sequence. The $\Sigma(\mathbf{d})$-graph and the subgraph induced by $\mathcal{CR}(\mathbf{d})$ are connected as a consequence of Taylor [4, 5]. For positive integers m and n with $0 \leq m \leq \binom{n}{2}$, let $\mathcal{G}(m, n)$ and $\mathcal{CG}(m, n)$ be the sets of all non-isomorphic graphs and the set of connected graphs of order n and size m, respectively. Let $G \in \mathcal{G}(m, n)$ with $e \in E(G)$ and $f \notin E(G)$. Define $G^{t(e,f)}$ to be a graph with $V(G^{t(e,f)}) = V(G)$ and $E(G^{t(e,f)}) = E(G - e + f)$. A transformation $t(e, f)$ is called an *edge jump.* Now let $\mathcal{T}(m, n)$ be a relation on $\mathcal{G}(m, n)$ defined by $(G, H) \in \mathcal{T}(m, n)$ if $G \not\cong H$ and H can be obtained from G by an edge jump. Since $\mathcal{T}(m, n)$ is symmetric, it follows that the $\mathcal{T}(m, n)$-graph is simple.

2. An intermediate value theorem

Let $\mathcal{G}$ be the class of all graphs. A graph parameter π is said to satisfy an *intermediate value theorem over a class of graphs* $\mathcal{J}$ if $G, H \in \mathcal{J}$ with $\pi(G) < \pi(H)$, then for every integer k with $\pi(G) \le k \le \pi(H)$ there is a graph $K \in \mathcal{J}$ such that $\pi(K) = k$. If a graph parameter π satisfies an intermediate value theorem over $\mathcal{J}$, then we write $(\pi, \mathcal{J}) \in$ IVT. Thus if $(\pi, \mathcal{J}) \in$ IVT, then $\{\pi(G) : G \in \mathcal{J})$ is uniquely determined by

$$\min(\pi, \mathcal{J}) := \min\{\pi(G) : G \in \mathcal{J}\} \text{ and } \max(\pi, \mathcal{J}) := \max\{\pi(G) : G \in J\}.$$

In 1964, Erdös and Gallai [6] proved that any regular graph on n vertices has chromatic number $k \le \frac{3n}{5}$ unless the graph is complete. Commenting on their result in a personal communication, Erdös wrote to Pullman "probably such a graph exists for every $k \le \frac{3n}{5}$, except possibly for trivial exceptional cases."

Caccetta and Pullman [7] confirmed and strengthened Erdös' conjecture by showing that if $k > 1$, then for every $n \ge \frac{5k}{3}$, there exists a connected, regular, k-chromatic graph of order n. This is an example an intermediate value theorem of χ over the class of all connected regular graphs of order n.

2.1 The Σ(d)-graphs

We will review in this subsection an intermediate value theorem on various graph parameters over $\mathcal{R}(\mathbf{d})$ and $\mathcal{CR}(\mathbf{d})$. We first prove a general result as follows:

Theorem 2.1 *Let $\mathcal{J} \subseteq \mathcal{R}(\mathbf{d})$ and the subgraph of $\Sigma(\mathbf{d})$-graph induced by J be connected. Let π be a graph parameter. For any graph G of degree sequence* **d** *and any switching σ, if* $|\pi(G) - \pi(G^{\sigma})| \le 1$, *then* $(\pi, \mathcal{J}) \in$ IVT.

Proof. Let $H, K \in \mathcal{J}$ such that $\pi(H) = \min\{\pi(G) : G \in \mathcal{J}\}$ and $\pi(K) = \max\{\pi(G) : G \in \mathcal{J}\}$. Since the subgraph of $\Sigma(\mathbf{d})$-graph induced by $\mathcal{J}$ is connected, there exists a path $P : H = G_1, G_2, \dots, G_t = K$ in $\mathcal{J}$. Thus there exists a sequence $\sigma_1, \sigma_2, \dots, \sigma_{t-1}$ such that $G_{i+1} = G_i^{\sigma_i}$. Since $|\pi(G_i) - \pi(G_{i+1})| = |\pi(G_i) - \pi(G_i^{\sigma_i})| \le 1$, it follows that $\{\pi(G_i) : i = 1, 2, \dots, t\} = \{k \in \mathbb{Z} : \pi(H) \le k \le \pi(K)\}$. Thus $(\pi, \mathcal{J}) \in$ IVT.

The following result can be obtained as consequences of Taylor [4, 5].

Corollary 2.2 *Let π be a graph parameter. For a graph G of degree sequence* **d** and *a switching σ, if* $|\pi(G) - \pi(G^{\sigma})| \le 1$, then $(\pi, \mathcal{R}(\mathbf{d})) \in$ IVT *and* $(\pi, \mathcal{CR}(\mathbf{d})) \in$ IVT.

We will now review an intermediate value theorem on several graph parameters over $\mathcal{R}(\mathbf{d})$ and $\mathcal{CR}(\mathbf{d})$. Here we use $\omega(G)$ and $\alpha(G)$ for the clique number and independent number of a graph G, respectively.

We proved in [8] and [9] the following result.

Theorem 2.3 *Let G be a graph and σ be a switching on G. If* $\pi \in \{\chi, \omega\}$, *then* $|\pi(G) - \pi(G^{\sigma})| \leq 1$.

Note that $\alpha(G) = \omega(\overline{G})$ for any graph G and $\overline{G}^{\sigma(a,b,c,d)} = \overline{G^{\sigma(a,b,c,d)}}$. Thus we have the following corollary.

Corollary 2.4 *Let G be a graph and σ be a switching on G. Then* $|\alpha(G) - \alpha(G^{\sigma})| \leq 1$.

For the matching number $\alpha'(G)$ of a graph G we obtained in [10] the following result.

Theorem 2.5 *If σ is a switching on G, then* $|\alpha'(G) - \alpha'(G^{\sigma})| \leq 1$.

The following results were obtained by Gallai [11] showing a relationship between the independence and covering number. Here we use $\beta(G)$ and $\beta'(G)$ for the covering and edge covering number of a graph G, respectively.

Theorem 2.6 *For a graph G of order n, $\alpha(G) + \beta(G) = n$.*

Theorem 2.7 *For a graph G of order n and $\delta \geq 1$. $\alpha'(G) + \beta'(G) = n$.*

As a consequence we obtain the following result.

Theorem 2.8 *Let G be a graph, $\delta(G) \geq 1$ and σ be a switching on G. If $\pi \in \{\beta, \beta'\}$, then* $|\pi(G) - \pi(G^{\sigma})| \leq 1$.

Let G be a graph and $F \subseteq V(G)$. Then F is called an *induced forest* of G if $G[F]$ contains no cycle. For a graph G, we define, $\mathrm{f}(G)$ as:

$$\mathrm{f}(G) := \max\{|F| : F \text{ is an induced forest in } G\}.$$

The graph parameter f is called the *forest number.* The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph G is known as the *decycling number* of G, and is denoted by $\phi(G)$: Thus for a graph G of order n, $\phi(G) + \mathrm{f}(G) = n$. We proved in [12] the following results on f and ϕ.

Theorem 2.9 *If S is any subset of vertices of G such that* $G[S]$ *is a forest, and σ is any switching on G, then* $G^{\sigma}[S]$ *contains at most one cycle.*

Proof. Let $S \subseteq V(G)$ and $G[S]$ contains no cycle. Let $a, b, c, d \in V(G)$ with $ab, cd \in E(G)$ and $ac, bd \notin E(G)$. Since $G[S]$ contains no cycle, it follows that $G[S] + ac$ and $G[S] + bd$ contains at most one cycle. Thus if $|S \cap \{a, b, c, d\}| \leq 3$, then $G^{\sigma}[S]$ contains at most one cycle. Now suppose that $\{a, b, c, d\} \subseteq S$. Since $G[S]$ is a forest, for any two vertices $u, v \in S$ there is at most one (u, v)-path in $G[S]$. In particular, if there is an (a, c)-path in $G[S]$, then there is no (b, d)-path in $G[S]$. Thus $G^{\sigma}[S]$ contains at most one cycle, where $\sigma = \sigma(a, b; c, d)$.

The following corollary can be obtained as a consequence of above theorem.

Corollary 2.10 *Let G be a graph and σ be a switching on G. If* $\pi \in \{\mathsf{f}, \phi\}$, then $|\pi(G) - \pi(G^{\sigma})| \le 1$.

A *dominating set* of a graph $G = (V, E)$ is a subset D of V such that each vertex of $V - D$ is adjacent to at least one vertex of D. The *domination number* $\gamma(G)$ of a graph G is the cardinality of a minimal dominating set with the least number of elements. We proved in [13] the following results.

Theorem 2.11 *If G is a graph with $\gamma(G) = \gamma$ and σ is a switching on G, then* $\gamma(G^{\sigma}) \le \gamma + 1$.

Proof. Let D be a minimum dominating set of G. Let $a, b, c, d \in V(G)$ with $ab, cd \in E(G)$ and $ac, bd \notin E(G)$. Put $\sigma = \sigma(a, b; c, d)$. If $\{a, b, c, d\} \cap D = \emptyset$ or $\{a, b, c, d\} \subseteq D$, then D is a dominating set of G^{σ}. If $a, b \in D$ or $c, d \in D$, then D is a dominating set of G^{σ}. Finally if $a \in D$ or $c \in D$, then $D \cup \{b\}$ or $D \cup \{d\}$ is a respective dominating set of G^{σ}. Thus $\gamma(G^{\sigma}) \le \gamma + 1$.

By the fact that a switching is symmetric we obtain the following result.

Corollary 2.12 *If σ is a switching on G, then* $|\gamma(G) - \gamma(G^{\sigma})| \le 1$.

Combining the results in this subsection we can conclude the following theorem.

Theorem 2.13 *Let* $\mathbf{d} = (d_1, d_2,...,d_n)$, $d_1 \ge d_2 \ge ... \ge d_n \ge 1$ *be a graphic degree sequence. Then* $(\pi, \mathcal{R}(\mathbf{d})) \in$ IVT *and* $(\pi, \mathcal{CR}(\mathbf{d})) \in$ IVT, *where* $\pi \in \{\chi, \omega, \mathsf{f}, \phi, \alpha, \alpha', \beta, \beta', \gamma\}$.

2.2 The $T(m, n)$-graphs

We recently proved in [14] that the $\mathcal{T}(m, n)$-graph and the subgraph of the $\mathcal{T}(m, n)$-graph induced by $\mathcal{CG}(m, n)$ are connected. We also obtained in the same paper the following results.

Theorem 2.14 *Let* $\pi \in \{\chi, \omega, \mathsf{f}, \phi, \alpha, \alpha', \beta, \beta', \gamma\}$ *Then for any* $G \in \mathcal{G}(m, n)$ *and an edge jump* $t(e, f)$ *on G,* $|\pi(G) - \pi(G^{t(e,f)})| \le 1$.

Theorem 2.15 *Let* $\pi \in \{\chi, \omega, \mathsf{f}, \phi, \alpha, \alpha', \beta, \beta', \gamma\}$ and $\mathcal{J} \in \{G(m, n), \mathcal{CG}(m, n)\}$. *Then* $(\pi, \mathcal{J}) \in$ IVT.

3. The extremal problems

An *extremal problem* asks for minimum and maximum values of a function $\pi : \mathcal{J} \Rightarrow \mathbb{Z}$. In our context we consider the problem of determining $\min(\pi, \mathcal{J})$ and $\max(\pi, \mathcal{J})$, where π is a graph parameter and $\mathcal{J}$ is a class of graphs. We emphasize on the graph parameters as stated in Section 2 and the classes of graphs $\mathcal{J} \in \{\mathcal{R}(r^n), \mathcal{CR}(r^n), \mathcal{G}(m, n), \mathcal{CG}(m, n)\}$. Therefore we use the following notation.

$$\min(\pi, r^n) = \min\{\pi(G) : G \in \mathcal{R}(r^n)\},$$
$$\max(\pi, r^n) = \max\{\pi(G) : G \in \mathcal{R}(r^n)\},$$
$$\mathrm{Min}(\pi, r^n) = \min\{\pi(G) : G \in \mathcal{CR}(r^n)\},$$
$$\mathrm{Max}(\pi, r^n) = \max\{\pi(G) : G \in \mathcal{CR}(r^n)\},$$
$$\min(\pi; m, n) = \min\{\pi(G) : G \in \mathcal{G}(m, n)\},$$
$$\max(\pi; m, n) = \max\{\pi(G) : G \in \mathcal{G}(m, n)\},$$
$$\mathrm{Min}(\pi; m, n) = \min\{\pi(G) : G \in \mathcal{CG}(m, n)\}, \text{ and}$$
$$\mathrm{Max}(\pi; m, n) = \max\{\pi(G) : G \in \mathcal{CG}(m, n)\}.$$

3.1 $\mathcal{R}(r^n)$ and $\mathcal{CR}(r^n)$

A classical result of Erdös and Gallai [6] gives a motivation to the extremal problem.

Theorem 3.1 *An r-regular graph G of order $n > r + 1$ has chromatic number $k \le \frac{3n}{5}$, with equality if and only if the complementary graph $\overline{G}$ of G is the union of disjoint 5-cycles.*

We obtained in [8] the extremal values of χ.

Theorem 3.2 *If $r \ge 2$ and $n \ge 2r$, then*

$$\min(\chi, r^n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 3.3 *If $r \ge 2$*, then

1. $\min(\chi, r^{r+1}) = \max(\chi, r^{r+1}) = r + 1$, *and*
2. $\min(\chi, r^{r+2}) = \max(\chi, r^{r+2}) = (r + 2)/2$.

Theorem 3.4 *For any $r \ge 4$ and odd integers such that $3 \le s \le r$, let q and t be integers satisfying $r + s = sq + t$, $0 \le t < s$. Then*

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q + 1 & \text{if } 1 \le t \le s - 2, \\ q + 2 & \text{if } t = s - 1. \end{cases}$$

Theorem 3.5 *For any even integer $r \geq 6$ and any even number s such that $4 \leq s \leq r$, let q and t be integers satisfying $r + s = sq + t$, $0 \leq t < s$. Then*

$$\min(\chi,\ r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q + 1 & \text{if } t \geq 2. \end{cases}$$

By using Brooks' theorem [15] and some graph construction we obtained the following theorems in [8].

Theorem 3.6 *Let $r \geq 2$. Then*

1. $\max(\chi,\ r^{2r}) = r$,
2. $\max(\chi,\ r^{2r+1}) = \begin{cases} 3 & \text{if } r = 2, \\ r & \text{if } r \geq 4, \end{cases}$
3. $\max(\chi,\ r^{n}) = r + 1$ *for* $n \geq 2r + 2$.

Theorem 3.7 *For any r and s such that $3 \leq s \leq r - 1$, we have*

1. $\max(\chi,\ r^{r+s}) \geq (r + s)/2$ *if $r + s$ is even, and*
2. $\max(\chi,\ r^{r+s}) \geq (r + s - 1)/2$ *if $r + s$ is odd.*

The exact values of $\max(\chi,\ r^{n})$ are not easy to obtain if $r + 3 \leq n \leq 2r - 1$. Result of Theorem 3.1 gives an upper bound for χ in the class of connected regular graphs of order n but the bound can be very far from the actual value depending on the regularity. We were able to improve the bound in [16] by introducing the notion of $F(j)$-graph.

Let j be a positive integer. An *$F(j)$-graph* is a $(j - 1)$-regular graph G of minimum order $f(j)$ with the property that $\chi(\overline{G}) > f(j)/2$. It is easy to see that $F(3)$-graph is C_5 and $f(3) = 5$. We found $F(j)$-graphs for all odd integers j as stated in the following theorems.

Theorem 3.8 *For odd integer $j \geq 3$, we have $f(j) = \frac{5}{2}(j - 1)$ if $j = 3 \pmod 4$ and $f(j) = 1 + \frac{5}{2}(j - 1)$ if $j \equiv 1 \pmod 4$.*

Theorem 3.9 [16] *Any r-regular graph of order n with $n - r = j$ odd and $j \geq 3$ has chromatic number at most $\frac{f(j) + 1}{2f(j)} \cdot n$, and this bound is achieved precisely for those graphs with complement equal to a disjoint union of $F(j)$-graphs.*

Problem 1. Find an $F(j)$-graph for even integer $j \geq 4$.

Problem 2. Find max(χ, r^{r+j}) if j is even and $4 \leq j \leq r-2$.

The extremal problem for ω has been completely answered in [9]. Since K_{r+1} is the only r-regular graph of order $r+1$, it follows that min(ω, r^{r+1}) = $r+1$. Given positive integers n and k with $k \leq n$, there exists a connected graph G of order n with $\omega(G) = k$. As we shall see in the next theorem that there is no regular graph G of order n having $\omega(G)$ strictly lies between $\frac{n}{2}$ and n.

Theorem 3.10 *Let* $\mathbf{d} = r^n$ *be a graphic degree sequence with* $r+2 \leq n \leq 2r+1$. *Then* max(ω, r^n) = $\lfloor \frac{n}{2} \rfloor$.

The idea of obtaining min(ω, r^n) is similar to what we have done for min(χ, r^n) and we have min(ω, r^n) = min(χ, r^n) in all situations.

Problem 3. We have obtained min(ω, r^n) and max(ω, r^n) in all situations. It is interesting to find Min(ω, r^n) and Max(ω, r^n).

Problem 4. By using the relation $\alpha(G) = \omega(\overline{G})$, can we obtain min($\alpha$, r^n), max(α, r^n), Min(α, r^n) and Max(α, r^n)?

For the graph parameter f, we found in [17] a lower bound of min(f, $\mathbf{d}$) by using the *probabilistic method.* In particular, we proved the following theorem.

Theorem 3.11 *Let G be a graph having degree sequence* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$. *Then*

$$\mathsf{f}(G) \geq 2 \sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

The value of min(f, r^n) is not easy to obtain if we work on r-regular graphs. It is reasonable to extend the class of r-regular graphs of order n to a larger class $\mathcal{G}_\Delta(n)$. Let n and Δ be positive integers with $n > \Delta$. Let $\mathcal{G}_\Delta(n)$ be the class of all graphs G of order n and $\Delta(G) = \Delta$. Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a sequence of non-negative integers. Define $\overline{\mathbf{d}}$ a degree sequence $(\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n})$, where $\overline{d_i} = n - d_i - 1$, for $i = 1, 2, \ldots, n$. It is clear that $\mathbf{d}$ is graphic if and only if $\overline{\mathbf{d}}$ is. We proved in [17] the following results.

Theorem 3.12 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ *be a graphic degree sequence and* $d_1 + 1 \leq n \leq 2d_1 + 1$. *Then*

1. $\min(\mathsf{f}, \mathbf{d}) = 2$ *if and only if* $d_1 = d_2 = d_3 = \cdots = d_n$ *and* $n = d_1 + 1$ *and*
2. *if* $\mathbf{d}$ *does not have a complete graph as its realization, then* $\min(\mathsf{f}, \mathbf{d}) = 3$ *if and only if* $\bar{\mathbf{d}}$ *has a disjoint union of stars as its realization.*

Theorem 3.13 *Let* $n = (\Delta + 1)q + t$, $0 \leq t \leq \Delta$. *Then*

1. $\min(\mathsf{f}, \mathcal{G}_\Delta(n)) = 2q$, *if* $t = 0$,
2. $\min(\mathsf{f}, \mathcal{G}_\Delta(n)) = 2q + 1$, *if* $t = 1$, *and*
3. $\min(\mathsf{f}, \mathcal{G}_\Delta(n)) = 2q + 2$, *if* $2 \leq t \leq \Delta$.

With some modification of Theorem 3.13 in the class of r-regular graphs of order n and some properties of $F(j)$-graph, we found $\min(\mathsf{f}, r^n)$ in all situations as stated in the following theorems in [18].

Theorem 3.14 *For* $r \geq 3$, *and* $n = r + j$, $1 \leq j \leq r + 1$

1. $\min(\mathsf{f}, r^n) = 2$, *if and only if* $n = r + 1$,
2. $\min(\mathsf{f}, r^n) = 3$, *if and only if* $n = r + 2$,
3. $\min(\mathsf{f}, r^n) = 4$, *for all even integers* n, $r + 3 \leq n$,
4. $\min(\mathsf{f}, r^n) = 4$, *for all odd integers* n, $r + 3 \leq n$ *and* $n \geq f(j)$,
5. $\min(\mathsf{f}, r^n) = 5$, *for all odd integers* n, $r + 3 \leq n$ *and* $n < f(j)$,

where $f(j) = \frac{5}{2}(j-1)$ *if* $j \equiv 3 \pmod 4$, *and* $f(j) = 1 + \frac{5}{2}(j-1)$ *if* $j \equiv 1 \pmod 4$.

Theorem 3.15 *For* $n \geq 2r + 2$ *and* $r \geq 3$, *write* $n = (r + 1)q + t$, $q \geq 2$ *and* $0 \leq t \leq r$. *Then*

1. $\min(\mathsf{f}, r^n) = 2q$ *if* $t = 0$,
2. $\min(\mathsf{f}, r^n) = 2q + 1$ *if* $t = 1$,
3. $\min(\mathsf{f}, r^n) = 2q + 2$ *if* $2 \leq t \leq r = 1$,
4. $\min(\mathsf{f}, r^n) = 2q + 3$ *if* $t = r$.

We obtained in [12] the values of $\max(\mathsf{f}, r^n)$, for all r and n as stated in the following theorems.

Theorem 3.16

$$\max(\mathsf{f}, r^n) = \begin{cases} n - r + 1 & \text{if } r + 1 \le n \le 2r - 1, \\ \left\lfloor \frac{nr-2}{2(r-1)} \right\rfloor & \text{if } n \ge 2r. \end{cases}$$

Note that if $r \ge 2$, then $\max(\mathsf{f}, r^n) = \mathrm{Max}(\mathsf{f}, r^n)$. The investigation of $\mathrm{Min}(\mathsf{f}, r^n)$ was considered in [19] and we settled almost all cases as stated in the following results.

Theorem 3.17 *Let n be an even integer $n \ge 12$. Then*

$$\mathrm{Min}(\mathsf{f}, 3^n) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & \text{if } n \equiv 2 \pmod 8, \\ \left\lfloor \frac{5}{8}n \right\rfloor & \text{otherwise.} \end{cases}$$

Theorem 3.18 *Let n and r be integers with $r \ge 4$. Then*

$$\mathrm{Min}(\mathsf{f}, r^n) \ge \left\lceil \frac{2n}{r} \right\rceil .$$

Let $n = rq + t$, $0 \le t \le r - 1$, $r \ge 4$. Then $\mathrm{Min}(\mathrm{f}, r^n) \ge 2q + \left\lceil \frac{2t}{r} \right\rceil$. By construction we have the following results.

$$\mathrm{Min}(\mathsf{f}, r^n) = \begin{cases} 2q & \text{if } t = 0, \\ 2q + 1 & \text{if } t = 1, 2, \\ 2q + 2 & \text{if } t > \frac{r}{2}. \end{cases}$$

Problem 5. Find $\mathrm{Min}(\mathrm{f}, r^n)$ if $3 \le t \le \frac{r}{2}$.

Let $\mathcal{B}(r^{2n})$ be the class of r-regular bipartite graphs of order $2n$. It was shown in [20], page 53 that the subgraph of the $\Sigma(r^{2n})$-graph induced by $\mathcal{B}(r^{2n})$ is connected. Therefore $(\mathsf{f}, \mathcal{B}(r^{2n})) \in$ IVT. We write $\min(\mathsf{f}, \mathcal{B}(r^{2n}))$ for $\min\{\mathsf{f}(G) : G \in \mathcal{B}(r^{2n})\}$ and $\max(\mathsf{f}, \mathcal{B}(r^{2n}))$ for $\max\{\mathsf{f}(G) : G \in \mathcal{B}(r^{2n})\}$. Thus $\mathsf{f}(\mathcal{B}(r^{2n}))$ is uniquely determined by $\min(\mathsf{f}, \mathcal{B}(r^{2n}))$, and $\max(\mathsf{f}, B(r^{2n}))$. Evidently, $\min(\mathsf{f}, \mathcal{B}(r^{2n})) = \max(\mathsf{f}, \mathcal{B}(r^{2n})) = 2n$ if $r \in \{0, 1\}$, $\max(\mathsf{f}, \mathcal{B}(2^{2n})) = 2n - 1$ and $\min(\mathsf{f}, B(2^{2n})) = \left\lceil \frac{3n}{2} \right\rceil$. We proved in [21] the following theorems.

Theorem 3.19 If $r \geq 2$, *then max*$(\mathsf{f}, \mathcal{B}(r^{2n})) = \max(\mathsf{f}, r^{2n}) = \lfloor \frac{nr-1}{r-1} \rfloor$.

Theorem 3.20 $\min(\mathsf{f}, \mathcal{B}(3^{2n})) = n + \lceil \frac{n}{4} \rceil$.

Theorem 3.21 $\min(\mathsf{f}, \mathcal{B}(4^{2n})) = n + \lceil \frac{n}{7} \rceil$.

The problem of determining $\min(\mathsf{f}, \mathcal{B}(r^{2n}))$ is not easy if $r \geq 5$.

Problem 6. Find $\min(\mathsf{f}, \mathcal{B}(r^{2n}))$ if $r \geq 5$.

Problem 7. Let $\mathcal{CB}(r^{2n})$ be the class of connected r-regular bipartite graphs of order 2^n and $r \geq 2$. It is clear that $\max(\mathsf{f}, \mathcal{CB}(r^{2n})) = \max(\mathsf{f}, \mathcal{B}(r^{2n}))$. Find $\min(\mathsf{f}, \mathcal{CB}(r^{2n}))$.

Problem 8. The hypercube Q_n is a connected n-regular bipartite graph of order 2^n. The exact values of $\mathsf{f}(Q_n)$ have been obtained when n is a power of 2. Details can be found in [22]. Find $\mathsf{f}(Q_n)$ for other values of n.

In [10], we determined the values of $\min(\alpha', r^n)$ and $\max(\alpha', r^n)$ for all r and n. Since $\min(\alpha', 0^n) = \max(\alpha', 0^n) = 0$ and $\min(\alpha', 1^{2n}) = \max(\alpha', 1^{2n}) = n$, we can assume that $r \geq 2$ and $n \geq r + 1$.

An existence of an r-regular Hamiltonian graph of order n implies that $\max(\alpha', r^n) = \lfloor \frac{n}{2} \rfloor$. A component of a graph is *odd* or *even* according as it has odd or even number of vertices. We denote by $o(G)$ the number of odd components of G. Tutte [23] proved the following theorem.

Theorem 3.22 *The number of edges in a maximum matching of a graph G is* $\frac{1}{2}(|V(G)|-d)$, *where* $d = \max_{S \subseteq V(G)}\{o(G - S) - |S|\}$.

Let $F(r, d)$ be the minimum order of an r-regular graph G with $\alpha'(G) = \frac{1}{2}(|V(G)|-d)$. It is clear that $|V(G)| \equiv d (\text{mod } 2)$. Wallis [24] found $F(r, 2)$ for all $r \geq 3$. More precisely, he proved the following theorem.

Theorem 3.23 *Let G be an r-regular graph with no 1-factor and no odd component. Then*

$$|V(G)| \geq \begin{cases} 3r + 7 & \text{if } r \text{ is odd, } r \geq 3, \\ 3r + 4 & \text{if } r \text{ is even, } r \geq 6, \\ 22 & \text{if } r = 4. \end{cases}$$

Furthermore, no such graphs exist for r = 1 or 2.

If G is an r-regular graph with $\alpha'(G) = \frac{1}{2}(|V(G)|-d)$, then there exists a k-subset K of $V(G)$ such that $o(G - K) = k + d$. If $k = 0$, then r is even, G contains d odd components, and each component of G has order at least $r + 1$. Suppose that $k \geq 1$ and $G - K$ has an odd component with p vertices where $p \leq r$. Thus the number of edges within the component is at most $\frac{1}{2}p(p - 1)$. This means that the sum of degrees of these p vertices in $G - K$ is at most $p(p - 1)$. But G is an r-regular graph, so the sum of degrees of these p vertices in G is pr. Hence the number of edges joining K to the component must be at least $pr - p(p - 1)$. For a fixed integer r and an integer p satisfying $1 \leq p \leq r$, the function $f(p) = pr - p(p - 1)$, $1 \leq p \leq r$ has minimum value $f(1) = f(r) = r$. So any odd component with r or less vertices is joined to K by r or more edges. Suppose that there are o_+ odd components of $G - K$ with more than r vertices and o_- odd components with less than or equal to r vertices. Thus

$$o_+ + o_- = k + d \tag{1}$$

$$o_+ + ro_- \leq kr. \tag{2}$$

From these 2 relations, we have $o_+ \geq \left\lceil \frac{rd}{r-1} \right\rceil = \mathrm{d} + \left\lceil \frac{d}{r-1} \right\rceil$ and $k \geq \left\lceil \frac{d}{r-1} \right\rceil$.

We obtained the following results in [10].

Theorem 3.24 *Let r be an even integer, $r \geq 2$. Then $F(r, d) = d(r + 1)$.*

Corollary 3.25 *Let r be an even integer, $r \geq 2$. If $n = (r + 1)d + e$, $0 \leq e \leq r$, then* $\min(\alpha', r^n) = \frac{dr}{2} + \left\lfloor \frac{1+e}{2} \right\rfloor$.

Suppose that r is odd and $r \geq 3$. Let G be an r-regular graph of order n such that $\alpha'(G) = \frac{1}{2}(n-d)$. Then d must be even. Put $d = 2q$. There exists a nonempty subset K of $V(G)$ of cardinality k such that $o(G - K) = k + 2q$. By (1) and (2), we have

$$n \geq k + (r+2)o_+ \geq \left\lceil \frac{2q}{r-1} \right\rceil + (r+2) + 2q + \left\lceil \frac{2q}{r-1} \right\rceil = \left\lceil \frac{2q}{r-1} \right\rceil (r+3) + 2q(r+2).$$

Wallis [24] defined $G(x, y)$ to be a graph with $x+y$ vertices, x and y being of degree $x+y-3$ and $x+y-2$, respectively. Thus $G(x, y)$ exists if and only if y is even and $y \geq 2$. It is noted that for any graph $G(x, y)$, it has y vertices of degree r and x vertices of degree $r-1$. Let x_i, y_i, $i = 1, 2, \ldots, m$, be integers such that $G(x_i, y_i)$ exists for all $i = 1, 2, \ldots, m$. We then construct a graph

$$G(x_1, y_1) * G(x_2, y_2) * \ldots * G(x_m, y_m)$$

from disjoint copies of the graphs by inserting a new vertex, say u, by joining u to all vertices of $G(x_i, y_i)$ which have the smallest degree, for $i = 1, 2, \dots, m$. With this notion we see that for an odd integer $r \geq 3$, $q = 1, 2, \dots, \frac{r-1}{2}$ and for any odd positive integers a_i, $i = 1, 2, \dots, 1 + 2q$ whose sum is r, it follows that

$$G_q = G(a_1, r + 2 - a_1) * G(a_2, r + 2 - a_2) * \dots * G(a_1 + 2q, r + 2 - a_1 + 2q)$$

is an r-regular graph on $(r + 2)(1 + 2q) + 1$ vertices with $\alpha'(G_q) = \frac{1}{2}(|V(G_q)| - 2q)$. We have the following results.

Theorem 3.26 *For an odd integer $r \geq 3$, then*

1. $F(r, 2q) = (r + 2)(1 + 2q) + 1$, *for* $q = 1, 2, \dots, \frac{r-1}{2}$,
2. *if* $q = \frac{r-1}{2}s + t, 0 \leq t \frac{r-1}{2}$, *then* $F(r, 2q) = sF(r, r-1) + F(r, 2t)$, *where* $F(r, 0) = 0$.

Corollary 3.27 *Let r be an odd integer, $r \geq 3$. If $F(r, 2q) \leq n < F(r, 2(q+1))$, then* $\min(\alpha', r^n) = \frac{1}{2}(n - 2q)$.

Problem 9. It is clear that $(\alpha', \mathcal{CR}(r^n)) \in$ IVT and it is easy to see that $\text{Max}(\alpha', r^n) = \lfloor \frac{n}{2} \rfloor$. Find $\text{Min}(\alpha', r^n)$.

3.2 $\mathcal{G}(m, n)$ **and** $\mathcal{CG}(m, n)$

We will discuss in this subsection the extremal problem for graph parameters over $\mathcal{G}(m, n)$ and $\mathcal{CG}(m, n)$.

Mantel's theorem [25] provides the maximum number of edges that a 2-chromatic graph of order n can have. On the other hand the minimum number of edges in a 2-chromatic graph of order $n \geq 2$ is 1 and the minimum number of edges in a 2-chromatic connected graph of order $n \geq 2$ is $n - 1$. Turán [26] extended the result of Mantel by introducing the *Turán graph*. This result of Turán is viewed as the origin of extremal graph theory. The *Turán graph* $T_{n,r}$ is the complete r-partite graph of order n whose partite sets differ in cardinality by at most 1.

Theorem 3.28 *Among the graphs of order n containing no complete subgraph of order* $r+1$, $T_{n,r}$ *has the maximum number of edges.*

In order to apply Turán's theorem in our context, we would like to state the following facts.

1. If $n = rq + t$, $0 \le t < r$, then $T_{n,r}$ consists of t partite sets of cardinality $\lfloor \frac{n}{r} \rfloor$ and $r-t$ partite sets of cardinality $\lfloor \frac{n}{r} \rfloor$.

2. Let $G \in G$(m, n). If $\omega(G) \le r$, then $m \le \varepsilon(T_{n,r})$.

3. $\varepsilon(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

4. Let $t(n, r) = \varepsilon(T_{n,r})$. Then for a fixed n, we get $t(n, r-1) < t(n,r)$ for all r, $2 \le r \le n$. In fact $t(n, r) - t(n, r-1) \ge \binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

We obtained in [14] the following theorems.

Theorem 3.29 *Let m, n and k be positive integers with* $n \ge k \ge 3$ *and* $\binom{k}{2} \le m < \binom{k+1}{2}$. *Then* $\max(\chi; m, n) = k$.

Theorem 3.30 *Let m, n and* $k \ge 2$ *be positive integers satisfying* $t(n, k-1) < m \le t(n, k)$. *Then* $\min(\chi; m, n) = k$.

We now conclude the following corollary.

Corollary 3.31 Let m, n and k be positive integers.

1. *If* $n \ge k$ *and* $\binom{k}{2} \le m < \binom{k+1}{2}$, *then* $\max(\omega; m, n) = k$.
2. *If* $t(n, k-1) < m \le t(n, k)$, *then* $\min(\omega; m, n) = k$.
3. *If* $t(n, k-1) < m \le t(n, k)$, *then* $\mathrm{Min}(\chi; m, n) = k$.
4. *If* $k \ge 3$ *and* $t(n, k-1) < m \le t(n, k)$, *then* $\mathrm{Min}(\omega; m, n) = k$.

Results on $\mathrm{Max}(\chi; m, n)$ and $\mathrm{Max}(\omega; m, n)$ can be obtained similarly as stated in the following theorems.

Theorem 3.32 *Let n, m and k be positive integers with* $n \ge k \ge 3$ *and* $\binom{k}{2} + n - k \le m < \binom{k+1}{2} + n - k - 1$. *Then* $\mathrm{Max}(\chi; m, n) = k$.

Theorem 3.33 *Let n,m and k be positive integers with* $n \geq k \geq 3$ *and* $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - k - 1$. *Then* Max(ω; *m, n*) = *k*.

Thus all extreme values of χ and ω over $\mathcal{G}(m, n)$ and $\mathcal{CG}(m, n)$ are obtained in all situations.

The extremal values of the graph parameter *f* over $\mathcal{G}(m, n)$ and $\mathcal{CG}(m, n)$ were obtained in [27].

Let *G* be a graph and *X, Y* be disjoint nonempty subsets of *V (G)*. Denote by $\varepsilon(X)$ the number of edges in $G[X]$ and $\varepsilon(X, Y)$ the number of edges in *G* connecting vertices in *X* to vertices in *Y*.

Let $G \in \mathcal{G}(m, n)$ and *F* be a maximum induced forest of *G*. Let $|F| = a$. Therefore $G - F$ has order $n - a$. An upper bound for m can be obtained by the following inequality.

$$m = \varepsilon(G - F) + \varepsilon(G - F, F) + \varepsilon(F) \leq \binom{n-a}{2} + a(n - a) + (a - 1).$$

Let $a = n - i$ for any $i \in \{1, 2,..., n - 2\}$. Then $m \leq (i + 1)n - \frac{i^2 + 3i + 2}{2}$

For an integer $i = 1, 2, \dots, n - 2$, let

$$\mathbf{M}_n(n - i) := (i + 1)n - \frac{i^2 + 3i + 2}{2}$$

It is clear that $\mathbf{M}_n(n-i)$ is an integer. We showed in [27] that max(f; *m, n*) = $n - i$ if and only if $\mathbf{M}_n(n-i+1) < m \leq \mathbf{M}_n(n-i)$ by constructing a graph $G \in \mathcal{G}(m, n)$ with $\mathbf{M}_n(n-i+1) < m \leq \mathbf{M}_n(n-i)$ and $f(G) = n - i$. Furthermore the graph *G* is connected. Therefore we have the following theorem.

Theorem 3.34 *Let n and m be integers satisfying* $0 < m \leq \binom{n}{2}$. *The* max(f; *m, n*) = $n - i$ *if and only if* $\mathbf{M}_n(n - i + 1) < m \leq \mathbf{M}_n(n - i)$ *and* Max(f; *m, n*) = $n - i$ *if and only if* $m \geq n - 1$ *and* $\mathbf{M}_n(n - i + 1) < m \leq \mathbf{M}_n(n - i)$.

In order to obtain the values of min(f; *m, n*), we first find the minimum number of edges of a graph of order *n* having the forest number *a*. Let $\mathcal{G}(n;\ f = a)$ be the set of graphs of order *n* having the forest number *a*. It is clear that $\mathcal{G}(n;\ f = a) \neq \emptyset$ if and only if $2 \leq a \leq n$. For integers *n* and *a*, let

$$\mathbf{m}_n(a) := \min\{\varepsilon(G) : G \in \mathcal{G}(n;\ f = a)\}:$$

Thus $\mathbf{m}_n(n) = 0$, $\mathbf{m}_n(n-1) = 3$ and $\mathbf{m}_n(2) = \binom{n}{2}$. It is easy to see that for a graph *G* of order $n \geq 2$, $f(G) = 2$ if and only if $G \cong K_n$. We now find $\mathbf{m}_n(a)$ for $2 < a < n$. Theorem 3.12 gives a characterization of graphs having forest number 3. Thus $\mathbf{m}_n(3) = \binom{n}{2} - n + 1$, for all $n \geq 4$. We proved in [27] the following lemma.

Lemma 3.35 *If G is a graph of order n with $\Delta(G) = \Delta$ and $f(G) = 2q + 1$ for some integer q, then $n \le (\Delta+1)q+1$.*

By Lemma 3.35, we have a lower bound for the maximum degree of a given graph in terms of its order and its forest number. In other words, if G is a graph of order n, then $\Delta(G) \ge \lceil \frac{2n}{f(G)} \rceil - 1$. In particular, if $f(G) = 2q$ for some integer q, then $\Delta(G) \ge \lceil \frac{n}{q} \rceil - 1$. By Lemma 3.35 the lower bound for $\Delta(G)$ can be improved if $f(G)$ is odd. That is, if $f(G) = 2q+1$ for some integer q, then $n \le (\Delta(G)+1)q + 1$ which is equivalent to $\Delta(G) \ge \lceil \frac{n-1}{q} \rceil - 1$. We have the following corollary.

Corollary 3.36 *Let G be a graph of order n and q be a positive integer. If $f(G) = 2q$, then* $\Delta(G) \ge \lceil \frac{n}{q} \rceil - 1$, and *if $f(G) = 2q + 1$, then* $\Delta(G) \ge \lceil \frac{n-1}{q} \rceil - 1$.

Let $\mathcal{G}^*(n;\ f = a) = \{G \in \mathcal{G}(n;\ f = a)$: G is a union of $\lceil \frac{a}{2} \rceil$ cliquesg$\}$. It is clear that $\mathcal{G}^*(n;\ f = a) \subseteq \mathcal{G}(n;\ f = a)$. We have the following theorem.

Theorem 3.37 *Let G be a graph of order n with $f(G) = a$. Then there exists a graph $H \in \mathcal{G}^*(n;\ f = a)$ such that $\varepsilon(H) \le \varepsilon(G)$.*

By Theorem 3.37, we know the structure of graphs of order n with prescribed the forest number. In general, for a graph $G \in \mathcal{G}(n;\ f = a)$, there may be many such graphs $H \in \mathcal{G}^*(n;\ f = a)$. We now seek for such a graph H with minimum number of edges.

By using the results of Mantel [25] and Turán [26] as mentioned in the previous subsection, we have the following results.

1. Let $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \cdots \cup p_kK_k$. Then the order of G is $p_1 + 3p_3 + 4p_4 + \cdots + kp_k$ and $f(G) = p_1 + 2(p_3 + p_4 + \cdots + p_k)$. Suppose that $p_1 \ge$, 2, $p_k \ge 1$ and $k \ge 4$. Then, by replacing $2K_1 \cup K_k$ by $K_3 \cup K_{k-1}$ we obtain a graph H with $\varepsilon(H) \le \varepsilon(G)$. Further, $\varepsilon(H) = \varepsilon(G)$ if and only if $k = 4$.

2. $\mathbf{m}_n(n-1) = 3$ if $n \ge 4$. Let $G \in \mathcal{G}(n;\ f = n-1)$. Then $\varepsilon(G) = 3$ if and only if $n \ge 4$ and $G = (n-3)K_1 \cup K_3$.

3. Let a be an integer with $\frac{2n}{3} \le a \le n - 1$. If (p, q) is the solution of $p + 3q = n$ and $p + 2q = a$, then $G = pK_1 \cup qK_3$ satisfies $f(G) = a$.

4. Let a be an integer with $\frac{2n}{3} \le a \le n - 2$. and $G \in \mathcal{G}(n;\ f = a)$ such that $\varepsilon(G) = \mathbf{m}_n(a)$. Then by Theorem 3.37, we can choose $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \cdots \cup p_kK_k \in \mathcal{G}^*(n;\ f = a)$ and $k \le 4$. If $k = 4$, then $p_1 \ge 2$. Thus, there exists a graph $H = p_1K_1 \cup pK_3$ such that $p+3q = n$, $p+2q = a$ and $\varepsilon(H) = \varepsilon(G) = \mathbf{m}_n(a)$.

5. Let a be an integer with $a < \frac{2n}{3}$ and $G \in \mathcal{G}(n;\ \mathsf{f} = a)$ and $\Delta(G) \geq 3$. Thus if $G = p_1K_1 \cup pK_3 \cup p_4K_4 \cup \cdots \cup p_kK_k \mathsf{f}(G) = a < \frac{2n}{3}$ and $\varepsilon(G) = \mathbf{m}_n(a)$, then $p_1 \leq 1$ and $k \geq 4$.

6. If $n = rq + t$, $0 \leq t < r$, then $T_{n,r}$ consists of t partite sets of cardinality $\lceil \frac{n}{r} \rceil$ and $r - t$ partite sets of cardinality $\lfloor \frac{n}{r} \rfloor$.

7. $\varepsilon(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

8. Let $t(n;\ r) = \varepsilon(T_{n,r})$. Then for a fixed n, by using elementary arithmetic, we get $t(n,\ r-1) < t(n,\ r)$ for all r, $2 \leq r \leq n$. In fact $t(n,\ r) - t(n,\ r-1) \geq \binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

Let $\bar{t}(n,\ r) = \binom{n}{2} - \varepsilon(T_{n,r})$. Summarizing the results, we have the following theorems.

Theorem 3.38 *Let n and a be integers with* $2 \leq a \leq n-1$*. Then*

1. $\mathbf{m}_n(n) = 0$,

2. $\mathbf{m}_n(n-1) = 3$ *if* $n \geq 3$ *and* $G = (n-3)K_1 \cup K_3$ *is the only graph of order n satisfying* $\mathsf{f}(G) = n-1$ *and* $\varepsilon(G) = 3$,

3. $\mathbf{m}_n(n-i) = 3i$ *if* $1 \leq i \leq \lceil \frac{n}{3} \rceil$,

4. *Suppose* $4 \leq a < \frac{2n}{3}$ *. Then* $\mathbf{m}_n(a) = \bar{t}(n,\ q)$ *if* $a = 2q$, and $\mathbf{m}_n(a) = \bar{t}(n-1,\ q)$ *if* $a = 2q+1$, *for some integer q, and*

5. $\mathbf{m}_n(3) = \binom{n-1}{2}$ *if* $n \geq 3$, and $\mathbf{m}_n(2) = \binom{n}{2}$ *if* $n \geq 2$.

Theorem 3.39 *Let n and m be integers with* $0 \leq m \leq \binom{n}{2}$*. Then*

1. $\min(\mathsf{f};\ m,\ n) = \max(\mathsf{f};\ m,\ n) = n$ *if and only if* $m \in \{0,\ 1,\ 2\}$,

2. $\min(\mathsf{f};\ m,\ n) = \max(\mathsf{f};\ m,\ n) = 2$ *if and only if* $m = \binom{n}{2}$, *and*

3. *for* $3 \leq a \leq n-1$, $\min(\mathsf{f};\ m,\ n) = a$ *if and only if* $\mathbf{m}_n(a) \leq \mathrm{m} < \mathbf{m}_n(a-1)$.

We now find the minimum number of edges of a connected graph order n having the forest number a. Let $\mathcal{CG}(n;\ f = a)$ be the set of all connected graphs of order n having the forest number a. For integers n and a, let

$$\mathbf{cm}_n(a) = \min\{\varepsilon(G) : G \in \mathcal{CG}(n;\ \mathsf{f} = a)\}.$$

Further, $\mathbf{cm}_n(n) = n-1$, $\mathbf{cm}_n(2) = \binom{n}{2}$. We now find $\mathbf{cm}_n(a)$ for $2 < a < n$.

Let $\mathcal{CG}^*(n;\ \mathsf{f} = a) = \{G \in \mathcal{CG}(n;\ \mathsf{f} = a) : G$ is obtained from $\lceil \frac{a}{2} \rceil$ disjoint cliques and $\lceil \frac{a}{2} \rceil - 1$ edgesg$\}$. We have the following theorem.

Theorem 3.40 *Let G be a connected graph of order n with* $\mathsf{f}(G) = a$. *Then there exists a graph* $H \in \mathcal{CG}^*(n;\ \mathsf{f} = a)$ *such that* $\varepsilon(H) \le \varepsilon(G)$.

By Theorem 3.40 we know that for a graph $G \in \mathcal{CG}(n;\ \mathsf{f} = a)$, there may be many such graphs $H \in \mathcal{CG}^*(n;\ \mathsf{f} = a)$. We now seek for such a graph H with minimum number of edges. By applying Turán Theorem once again, we have the following theorems.

Theorem 3.41 *Let n and a be integers with* $2 \le a \le n-1$. *Then*

1. $\mathbf{cm}_n(n) = n-1$,
2. *Suppose that* $4 \le a \le n-1$. *Then* $\mathbf{cm}_n(a) = \bar{t}(n,\ q) + q - 1$ *if* $a = 2q$, *and* $\mathbf{cm}_n(a) = \bar{t}(n-1,\ q) + q$ *if* $a = 2q+1$, *for some integer q, and*
3. $\mathbf{cm}_n(3) = \binom{n-1}{2} + 1$ *if* $n \ge 3$, *and* $\mathbf{cm}_n(2) = \binom{n}{2}$ *if* $n \ge 2$.

Theorem 3.42 *Let n and m be integers with* $n-1 \le m \le \binom{n}{2}$. *Then*

1. $\mathrm{Min}(\mathsf{f};\ m,\ n) = \mathrm{Max}(\mathsf{f};\ m,\ n) = n$ *if and only if* $m = n-1$,
2. $\mathrm{Min}(\mathsf{f};\ m,\ n) = \mathrm{Max}(\mathsf{f};\ m,\ n) = 2$ *if and only if* $m = \binom{n}{2}$, *and*
3. *for* $3 \le a \le n-1$, $\mathrm{Min}(\mathsf{f};\ m,\ n) = a$ *if and only if* $\mathbf{cm}_n(a) \le m < \mathbf{cm}_n(a-1)$.

Problem 10. Several graph parameters have been proved to satisfy an intermediate value theorem over $\mathcal{G}(m,\ n)$ and $\mathcal{CG}(m,\ n)$ as stated in Theorem 2.15. Find $\min(\pi;\ m,\ n)$; $\max(\pi;\ m,\ n)$, $\mathrm{Min}(\pi;\ m,\ n)$ and $\mathrm{Max}(\pi;\ m,\ n)$ where $\pi \in \{\alpha,\ \alpha'\ \gamma\}$.

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