บทความวิชาการ

ความน่าจะเป็นของการคงอยู่ในระบบพลวัติแบบสุ่ม

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บทคัดย่อ

บทความนี้เป็นการศึกษาทบทวนการคงอยู่ในกระบวนการพลวัติแบบสุ่มในทางทฤษฎี ทางการจำลองทางคอมพิวเตอร์ และทางการทดลอง โดยมีการวิเคราะห์ทางทฤษฎีเกี่ยวกับความน่าจะเป็นของการคงอยู่ของกระบวนการอย่างง่าย ค่าความน่าจะเป็นของการคงอยู่ที่คำนวณได้มีการลดลงตามเวลาแบบกฎการยกกำลัง ค่าของเลขชี้กำลังการคงอยู่ในกระบวนการต่างๆ ได้ถูกคำนวณทางทฤษฎี และมีการใช้การประมาณค่า เช่น การประมาณช่วงเวลาอิสระ (IIA) ในกระบวนการแบบราบเรียบ (smooth process) เพื่อให้สามารถหาค่าของเลขชี้กำลังการคงอยู่ได้ ในการศึกษาความผันผวนระหว่างพื้นผิว (fluctuating interfaces) มีการวิเคราะห์ความสัมพันธ์ระหว่างเลขชี้กำลังการเติบโตและเลขชี้กำลังการคงอยู่ ผลการจำลองทางคอมพิวเตอร์และผลการทดลองที่เกี่ยวข้องได้รับการอภิปราย

คำสำคัญ: ความน่าจะเป็นของการคงอยู่ เลขชี้กำลังของการคงอยู่ กระบวนการแบบสุ่ม

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Persistence Probability in Stochastic Dynamical Systems

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ABSTRACT

Theoretical, simulation and experimental studies of persistence in stochastic dynamical processes are reviewed. Persistence probabilities of simple processes are analyzed theoretically. The calculated persistence probability displays power-law decrease with time. The associated persistence exponent is calculated. Some approximations are introduced to obtain nontrivial value of the exponent. Independent interval approximation (IIA) is introduced in the smooth process. The relation between the growth exponent and the persistence exponent of fluctuating interfaces is analyzed. The associated simulation and experimental results are also discussed.

Keywords: persistence probability, persistence exponent, stochastic process

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Introduction

Stochastic dynamical system is a system that contains randomness in its process. The randomness causes fluctuation in the dynamical process. That fluctuation, in turn, causes disorderliness in the system. However, we can predict how a random system will evolve by studying statistical properties of the system. One important quantity is the persistence probability which is the main quantity studied in this work. The persistence probability is the probability that the sign of a stochastic variable persists in time or the probability that the value of that variable remains greater or smaller than the initial value (determined at a pre-specified time). In other words, the persistence probability is the probability that the value of a stochastic variable does not return to the initial one throughout a considered time interval.

The idea of "persistence" is developed and widely used to describe statistics of many stochastic processes, for example, the random walker [1-2], the diffusion particles [1, 3], the spin system [4] and many body nonequilibrium systems [5]. The persistence concept also helps predict asymptotic properties of growth processes [6-10]. In addition, persistence concept is applied to the study of population dynamics in biology [11]. In experiments, persistence probability is an important quantity measured in fabrication of electronic/nanoscale devices [12-15]. Not only the persistence concept is used in various scientific researches, but it is also helpful for prediction of stock market fluctuations [16]. With previous data, the persistence probability of a stock price can help predict the volatility of stock price.

The persistence probability, $P(t)$, in many systems [1-16] shows power-law behavior with time as $P(t) : t^{-\theta}$, where θ is the persistence exponent whose value is system-dependent. In this work, we introduce some simple stochastic processes and the theoretical calculation of their persistence probabilities and exponents. Some processes are studied simulationally and experimentally and the obtained persistence exponents are compared to those in theory.

Theory of persistence probability of simple processes

Normally, the equation of motion of any stochastic dynamical system consists of a deterministic term and a random or noise term. The general equation of motion in d dimensions can be written as

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t} = F(\phi,\mathbf{x},t) + \eta(\mathbf{x},t) \quad (1)$$

where $\phi(\mathbf{x},t)$ is a stochastic variable which is a function of position $\mathbf{x} = (x_1, x_2, ..., x_d)$ and time t. $F(\phi,\mathbf{x},t)$ is the slowly varying part whose form depends on behaviors of the system while $\eta(\mathbf{x},t)$ is the fluctuating or the noise part whose correlation also depends on the system. In this section, we review some stochastic dynamical systems and their associated persistence probabilities and exponents theoretically.

1. One dimensional random walker (Brownian motion in one dimensional system)

In this system, one walker is moving in one dimension. The stochastic variable $\phi(t)$ is the position of the walker at time t. The persistence probability of a random walker is the probability that the walker does not return to or cross its original position throughout the time interval t. Due to the random behavior of the walker, its equation of motion can be written as

$$\frac{\partial\phi(t)}{\partial t}=\eta(t), \tag{2}$$

where $\eta(t)$ is the uncorrelated random Gaussian noise or white noise. This kind of noise has zero mean: $\langle\eta(t)\rangle=0$ and its correlation function is the delta function: $\langle\eta(t)\eta(t')\rangle=\delta(t-t')$. From eq.(2) we can write $\phi(t)=\int_0^t\eta(t'')dt''$ and $\phi(t')=\int_0^{t'}\eta(t''')dt'''$. Then the correlation yields

$$\langle\phi(t)\phi(t')\rangle=\left\langle\int_0^t\eta(t'')dt''\int_0^{t'}\eta(t''')dt'''\right\rangle=\int_0^t dt''\int_0^{t'}dt'''\langle\eta(t'')\eta(t''')\rangle=\int_0^t dt''\int_0^{t'}dt'''\delta(t''-t''')=\min(t,t'). \tag{3}$$

It can be seen that the correlation depends on two parameters: both t and t', so $\phi(t)$ is a non-stationary Gaussian process. The stationary Gaussian process is the process whose joint probability distribution does not change with time i.e. the distribution does not depend directly on a specific time instant, but may depend on the time difference. So it is more convenient to transform $\phi(t)$ to a variable that is stationary in order that the correlation depends on only one parameter: the time difference $|t-t'|$. A normalized stationary variable $X(t)$ is defined as $X(t)=\phi(t)\big/\sqrt{\langle\phi^2(t)\rangle}$. From $\langle\varphi^2(t)\rangle=\int_0^t dt''\int_0^t dt'''\delta(t''-t''')=t$, the correlation function of $X(t)$ can be written as $\langle X(t)X(t')\rangle=\min(t,t')\big/\sqrt{tt'}$, which still depends on t and t'. The new time parameter $T\equiv\ln t$ [3] is then defined so that the correlation $\langle X(T)X(T')\rangle$ depends on only one parameter which is $|T-T'|$:

$$\langle X(T)X(T')\rangle=\frac{\min(e^T,e^{T'})}{\sqrt{e^T\cdot e^{T'}}}=e^{-|T-T'|/2}. \tag{4}$$

The correlation can then be written as a function f where $f(T)=e^{-T/2}$. The reason to have the minus sign is because, conventionally, the value of the coefficient is positive. It has been proved in literature [17] that the process that has purely exponential correlator has an exact solution. This kind of process is called the Markovian process. As a result, the random walker process, in which the correlation is in the exponential form: $f(T)=e^{-T/2}$, is Markovian and has an exact solution.

For the stationary Gaussian process, it has been shown [17] that if $f(T)$ of a system in d dimensions goes to 0 faster than $1/T^d$ (or $f(T) < 1/T^d$) for large T, the persistence probability of the system must be in the exponential form as $P(T) : e^{-\mu T}$. For the one dimensional random walker ($d = 1$), $f(T) = e^{-T/2}$ decays faster than $1/T$ for large T, so $P(T) : e^{-\mu T}$ where μ = 1/2 in this case. From $T \equiv \ln t$, the probability as a function of time t is $P(t) : t^{-1/2}$ which means the persistence exponent of one dimensional Brownian motion is θ = 1/2.

2. Diffusion process

The motion of particles in diffusion can be expressed by the diffusion equation in d dimensions as

$$\frac{\partial \phi(\mathbf{x},t)}{\partial t} = \nabla^2 \phi(\mathbf{x},t), \tag{5}$$

where $\nabla^2 = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$. The variable $\phi(\mathbf{x},t)$ represents the density fluctuation of the diffusing particles. The random term is not presented in the equation, but it is in the initial conditions as $\langle \phi(\mathbf{x},0)\phi(\mathbf{x}',0) \rangle = \delta^d(\mathbf{x}-\mathbf{x}')$ (where $\langle \phi(\mathbf{x},0) \rangle = 0$). The initial conditions state that the density fluctuations of different positions are uncorrelated. The solution of the diffusion equation is $\phi(\mathbf{x},t) = \int d^d x' G(\mathbf{x}-\mathbf{x}',t)\phi(\mathbf{x}',0)$, where $G(\mathbf{x},t) = \left[1/(4\pi t)^{d/2}\right] e^{-|\mathbf{x}|^2/4t}$ is the Green function. During a specific time interval, it is possible that $\phi(\mathbf{x},t)$ at a particular $\mathbf{x}$ crosses the initial value ϕ_0 several times. The density of ϕ_0-crossings which is the number of time when $\phi(\mathbf{x},t) = \phi_0$ per unit time is denoted by ρ. The persistence probability is the probability that the density fluctuation $\phi(\mathbf{x},t)$ does not change sign up to the time t, or the probability that $\phi(\mathbf{x},t)$ at a particular $\mathbf{x}$ does not return to the initial value $\phi(\mathbf{x},0) \equiv \phi_0$ up to the time t. To find the persistence probability, the stationary Gaussian variable is introduced: $X(t) = \phi(\mathbf{x},t) / \sqrt{\langle \phi^2(\mathbf{x},t) \rangle}$. The correlation function $\langle X(t)X(t') \rangle = \left(4tt'/(t+t')^2\right)^{d/2}$ can be written in terms of $T \equiv \ln t$ as

$$\langle X(T)X(T') \rangle = \left[\frac{2\sqrt{e^T \cdot e^{T'}}}{e^T + e^{T'}}\right]^{d/2} = \left[\frac{2}{e^{\left(\frac{T-T'}{2}\right)} + e^{\left(\frac{T'-T}{2}\right)}}\right]^{d/2} = \left[\mathrm{sech}\left((T-T')/2\right)\right]^{d/2}. \tag{6}$$

The correlation can be written as a function f where $f(T) = \left[\mathrm{sech}(T/2)\right]^{d/2}$ which is not exponential, so the process is a non-Markovian process and does not have an exact solution. Even though the system is non-Markovian, the function $f(T)$ decreases faster than $1/T^d$ as $T \to \infty$. As a result, the persistence probability scales exponentially as $P(T) : e^{-\theta T}$

or scales as a power-law with time t as $P(t) : t^{-\theta}$. The persistence exponent θ depends on the dimension of the system and cannot be calculated exactly.

The correlation function of the stochastic process, in general, is in the form $f(T) \approx 1-aT^{\alpha}$, where 0 < $\alpha \leq$ 2 for small T [17]. For α = 2, the process is called "smooth" [18] which means that the density of ϕ_0-crossings, ρ, is finite and depends on the second derivative of $f(T=0)$ as $\rho = \sqrt{-f''(0)}/\pi$ [18]. The ϕ_0-crossing time is uniformly distributed over a specified time interval. For α < 2, ρ is not defined, giving infinite density so the persistence exponent cannot be computed theoretically. For the diffusion process, the correlation for small T is $f(T) \approx 1-\left(d/16\right)T^2$, which is "smooth" indicated by α = 2. Majumdar and coworkers [3] suggest the "independent interval approximation (IIA)", one method to find the persistence exponent. In this method, the time intervals between ϕ_0-crossings are assumed to be uncorrelated. Because ϕ is the density fluctuation, we consider $\phi_0 = 0$ (so $X_0 = 0$) and use IIA to find the persistence probability. The new variable $\sigma = \mathrm{sgn}(X)$ [1] is defined to vary between +1 and -1 as X crosses zero. The correlation $A(T) \equiv \langle \sigma(0)\sigma(T) \rangle = \left(2/\pi\right)\sin^{-1}\left(f(T)\right)$ can be written in terms of the probability of n zero-crossing, $P_n(T)$, which is the probability that X crosses zero n times within the interval T, as $A(T) = \sum_{n=0}^{\infty}(-1)^n P_n(T)$. $P_n(T)$ can be expressed in terms of $p(T)$ which is the distribution of the intervals between zeros and $Q(T)$ which is the probability that there are no zeros outside the interval T as

$$P_n(T) = \frac{1}{\langle T \rangle}\int_0^T dT_1 Q(T_1)\int_{T_1}^T dT_2 p(T_2 - T_1)\int_{T_2}^T dT_3 p(T_3 - T_2)\cdots\int_{T_{n-1}}^T dT_n p(T_n - T_{n-1})Q(T - T_n). \quad (7)$$

where $\langle T \rangle = 1/\rho$ is the mean interval size between zeros. The probability of n crossings between 0 and T, $P_n(T)$, can be written as the product of the distribution of individual crossing, $p(T)$. It can be seen that the assumption of IIA helps relate the distribution of n zero-crossing intervals to the distribution of independent single intervals. After taking the Laplace transform, $\tilde{P}_n(s) = \int_0^{\infty} e^{-sT}P_n(T)dT$, with the condition that the total probability $\sum_{n=0}^{\infty}P_n(T) = 1$ and, by definition, $p(T) = -dQ(T)/dT$, the interval distribution can be written as [3]

$$\tilde{p}(s) = \frac{\left(2-F(s)\right)}{F(s)} \quad (8)$$

where

$$F(s) = 1 + \frac{s}{2\rho}\left(1 - s\tilde{A}(s)\right). \quad (9)$$

Here $\tilde{A}(s)$ is the Laplace transform of $A(T)$. Because the persistence probability $P_n(T)$ of no zeros within the interval is expected to be $P_{n=0}(T) : e^{-\theta T}$ we obtain $\tilde{P}_0(s) : 1/\left(s+\theta\right)$ that has a simple pole at $s=-\theta$. From the definition of the interval distribution, the relation $p(T) : e^{-\theta T}$ is expected as well. From eq. (8), it can be concluded that θ is determined by the condition $F(s)=0$. After replacing all functions and solving eq. (9) numerically, the persistence exponents in one, two and three dimensions are 0.1203, 0.1862, and 0.2358 respectively [3].

In general, IIA method is developed to be used in a smooth process with nonzero M-crossings in Refs. [19-20].

3. Fluctuating interfaces

In research of dynamics of surface growth processes, there are fluctuations in the height profile $H(\mathbf{x},t)$ of each site on the substrate due to the randomness in deposition process. The height fluctuation of the growing interfaces $h(\mathbf{x},t) \equiv H(\mathbf{x},t) - \bar{H}(\mathbf{x},t)$ can be described by the continuum equation [6]

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = -\left(\nabla^2\right)^{z/2} h(\mathbf{x},t) + \eta(\mathbf{x},t), \qquad (10)$$

where z is the dynamical exponent that characterizes how the saturation time depends on the system size L. The exponent z depends on the dimension of the system and the universality class of the model. The growth models that are in the same universality class have the same scaling behaviors in the asymptotic limit. The persistence probability of the growth process is the probability that $h(\mathbf{x},t)$ does not cross the initial value throughout the time interval t.

The simplest growth model is the random deposition (RD) model [21]. The deposited atom falls directly on the substrate at a random site and sticks at that site. The deposited atom cannot move after landing so there is no correlation between neighbors. The associated continuum equation is $\partial h(\mathbf{x},t)/\partial t = \eta(\mathbf{x},t)$ which is the same equation as that of a random walker. As a result, the dynamical process of the RD model is Markovian in which the persistence probability has a power-law decay $P(t) : t^{-\theta}$ where θ = 1/2.

For other growth processes, the deposited atom is allowed to diffuse along the substrate by a model-dependent diffusion rule. By using the same procedure, starting from the beginning of the growth process where $h(\mathbf{x},0)=0$, the correlation function of the stationary Gaussian variable $X(t) = h(\mathbf{x},t)\Big/\sqrt{\langle h^2(\mathbf{x},t)\rangle}$ and time T is

$$f_0(T) = \left(\cosh\left(\frac{T}{2}\right)\right)^{2\beta} - \left|\sinh\left(\frac{T}{2}\right)\right|^{2\beta}, \qquad (11)$$

where β is the growth exponent, the second critical exponent that characterizes the time dependence of the interface width during the transient stage. In general, $0 \leq \beta < 1$. For uncorrelated white noise, β = (1/2)(1 - *d/z*) [6, 22]. For the steady-state when the growth time is greater than L^z, the values of the initial height $h(\mathbf{x}, t_0)$ at the initial time t_0 are not necessary zero. The correlation function of the height difference, $h(\mathbf{x}, t+t_0) - h(\mathbf{x}, t_0)$, is

$$f_S(T) = \cosh(\beta T) - \frac{1}{2}\left|2\sinh\left(\frac{T}{2}\right)\right|^{2\beta}. \tag{12}$$

The deterministic term in the equation indicates the correlation in surface growth due to interactions between neighbors. These interactions cause the dynamics non-Markovian as can be seen from the fact that the correlation functions in eqs.(11)-(12) are not purely exponential. In the limit $T \to \infty$, $f_0(T) : e^{-(1-\beta)T}$ and $f_S(T) : e^{-[\min(\beta, 1-\beta)]T}$ which both decrease faster than $1/T^d$. The persistence probability, therefore, scales as $P(T) : e^{-\theta T}$ or $P(t) : t^{-\theta}$. θ depends on the dimension of the system and the universality class and, in this case, cannot be calculated exactly. Note that for a model that has $\beta = 1/2$ such as the RD model, the model has the same correlation function as a random walker, $f_{0,S} : e^{-T/2}$, which is Markovian.

In the limit $T \to 0$, both correlation functions can be written as $f(T) \approx 1 - O|T|^{\alpha}$, where $\alpha = 2\beta,\ \beta < 1$. The results give $\alpha < 2$ so the processes in both transient stage and steady-state are non-smooth processes which cannot be solved by IIA method.

For a surface growth process that is not far from the Markovian process ($\beta = 1/2$ and $\alpha = 1$), Krug and coworkers [6] apply the perturbation theory to calculate θ. For a model that has $\beta = 1/2 + \varepsilon$ where ε is a small parameter, the perturbation to the exact Markovian process can be presented in the form of the correlation function that differs slightly from the exponential one: $f(T) = e^{-|T|/2} + \varepsilon\varphi(|T|) + O(\varepsilon^2)$. The persistence exponents are then found to be [6]

$$\theta^0 = \frac{1}{2} - \varepsilon\left(2\sqrt{2} - 1\right) + O\left(\varepsilon^2\right), \tag{13}$$

$$\theta^S = \frac{1}{2} - \varepsilon + O\left(\varepsilon^2\right). \tag{14}$$

For any process that satisfies eq. (10) (including the process that is far from Markovian), Krug and coworkers use the theorem for the stationary Gaussian processes proved by Stepian [17] to show that the persistence exponent is related to the growth exponent of the process by $\theta^S = 1 - \beta$ [6] which supports eq. (14). However, this relation is satisfied only in the process that has up-down symmetry in h (the interface fluctuation is invariant under $h \to -h$). Constantin and coworkers [9] generalize this relation to the process without this symmetry. The incremental autocorrelation function for the height fluctuation in the steady-state is

$$C(t,t') = \lim_{t_0 \to \infty} \left\langle \left[h(\mathbf{x}, t+t_0) - h(\mathbf{x}, t'+t_0) \right]^2 \right\rangle. \tag{15}$$

For any growth process, h is a self-affine field i.e. h is invariant under anisotropic transformation: $h(x) : b^{-a} h(bx)$. The incremental autocorrelation function of self-affine field does not depend on a specific time, but the time difference no matter h is Gaussian or not i.e. $C(t,t') : |t-t'|^{2\beta}$ for large $|t-t'|$. The new variable $Y(\mathbf{x},t) = h(\mathbf{x},t+t_0) - h(\mathbf{x},t_0)$ is defined. The incremental autocorrelation function of $Y(\mathbf{x},t)$ is the same as that of $h(\mathbf{x},t)$. The probability that $h(\mathbf{x},t)$ at the growth time t crosses the initial value $h(\mathbf{x},t_0)$ (the value at the initial time t_0) is the probability that $Y(\mathbf{x},t)$ crosses zero. Starting at time t_0 where $Y(t_0) = 0$ for all $\mathbf{x}$, $Y(\tau) = h(\tau + t_0) - h(t_0)$ at time $t_0 + \tau$. $P(Y,\tau)$ is defined as the probability that the field has value Y (not necessary for the first time) at time $t_0 + \tau$. The scaling form of $P(Y,\tau)$ should be [9]

$$P(Y,\tau) = \frac{1}{\sqrt{\langle Y^2(\tau) \rangle}} f\left(\frac{Y}{\sqrt{\langle Y^2(\tau) \rangle}} \right), \tag{16}$$

where

$$f(u) = \begin{cases} \text{const.} & u = 0 \\ 0 & u \to \pm\infty \end{cases} . \tag{17}$$

By using eq. (15), $\sqrt{\langle Y^2(\tau) \rangle} : \sqrt{\tau^{2\beta}} = \tau^{\beta}$ for large τ. The probability that the field crosses zero at time $t_0 + \tau$ is $P(0,\tau) : 1/\sqrt{\langle Y^2(\tau) \rangle} = \tau^{-\beta}$. $P(0,\tau)$ decreases as the time interval τ increases, thus $\rho(\tau) \equiv P(0,\tau)$ is the density of zero-crossings during time $\tau \to \tau + d\tau$. Let T be the total time range and $N(T)$ be the total number of zero-crossings or the total number of time intervals. For large T, one can write

$$N(T) = \int_0^T \rho(\tau) d\tau = \int_0^T \tau^{-\beta} d\tau \approx T^{1-\beta}. \tag{18}$$

For the up-down asymmetric process, the behavior of the field Y above and under zero is different. The persistence probability is then separated into two types i.e. the probability that Y remains greater than zero, P_+^S, and the probability that Y remains smaller than zero, P_-^S, where $P_\pm^S(\tau) : \tau^{-\theta_\pm^S}$. Let $N_+(N_-)$ be the total number of + (–) intervals, so $N_\pm = N/2$. Let $n_+(n_-)$ be the number of + (–) intervals of length τ within the time interval [$t_0, t_0 + T$]. One obtains [9] $n_\pm(\tau) / N_\pm(\tau) = q_\pm(\tau)$, where $q_\pm(\tau) \equiv -dP_\pm^S / d\tau : \tau^{-(1+\theta_\pm^S)}$ is the distribution of $\pm$ intervals. Therefore $n_\pm : N(T) \tau^{-(1+\theta_\pm^S)}$. The total time range is

$$T = \int_0^T \tau\left(n_+(\tau)+n_-(\tau)\right)d\tau = N(T)\int_0^T \tau\left(\tau^{-\left(1+\theta_+^S\right)}+\tau^{-\left(1+\theta_-^S\right)}\right)d\tau$$
$$T^{\beta} : \frac{T^{\left(1-\theta_+^S\right)}}{\left(1-\theta_+^S\right)}+\frac{T^{\left(1-\theta_-^S\right)}}{\left(1-\theta_-^S\right)}. \quad (19)$$

For $T \to \infty$ and $\theta_{\pm}^S \le 1$, $T^{\beta} : T^{1-\theta_+^S}$ if $\theta_+^S < \theta_-^S$ and $T^{\beta} : T^{1-\theta_-^S}$ if $\theta_+^S > \theta_-^S$. As a result, the general relation of the persistence exponent and the growth exponent is

$$\beta = \max\left[1-\theta_+^S, 1-\theta_-^S\right]. \quad (20)$$

Besides, the continuum equation in eq. (10) and the following theoretical calculations for θ are also used to describe the step fluctuation (fluctuation in the direction parallel to the substrate) on a tilted monatomic step surface [10]. Constantin and coworkers study the fluctuation in the step position on the vicinal surface (the tilted surface that consists of terraces separated by monoatomic steps). One of their researches is on the fluctuating step occurred by atoms moving in and out of step edges at high temperature. They found that the fluctuation can be described by the linear Langevin equation with a non-conserved noise and the associated persistence exponent is θ = 0.34.

Simulation of persistence probability

The persistence probabilities of all stochastic processes described in the previous section are found, via simulation work, to have power-law decay in time. The results are in agreement with those calculated analytically. Moreover, simulation is an easy way to obtain the persistence exponents of complicated processes, such as non-smooth/non-Markovian processes.

For the simple diffusion process, Majumdar and coworkers [3] show the simulation results of θ extracted from a least-squares fit of the double-log plots of P vs t in one, two, and three dimensions to be 0.1207 ± 0.0005, 0.1875 ± 0.0010 and 0.2380 ± 0.0015, respectively. The errors are obtained from the least-squares fits. The results are in accordant with analytic calculation.

From our simulation study of fluctuations in the growth processes, the persistence probability P of the height fluctuation in the growth direction of the film grown on a flat initial substrate is found to decay as a power-law with time. Fig.1 shows the persistence probability of the RD model, which is a Markovian process. The unit of time in the plot is monolayer (MLs) which is discrete. One monolayer is the time when the number of atoms deposited on the surface equals the total number of atoms in one layer. The growth exponent of the RD model, extracted from the plot of the interface width (W) as a function of time, is $\beta \approx$ 0.50. The persistence exponent obtained from the slope of P is θ = 0.499 ± 0.001 which is consistent with that of the Markovian process: θ = 0.5. Besides, the results confirm theoretical relation θ = 1 - β.

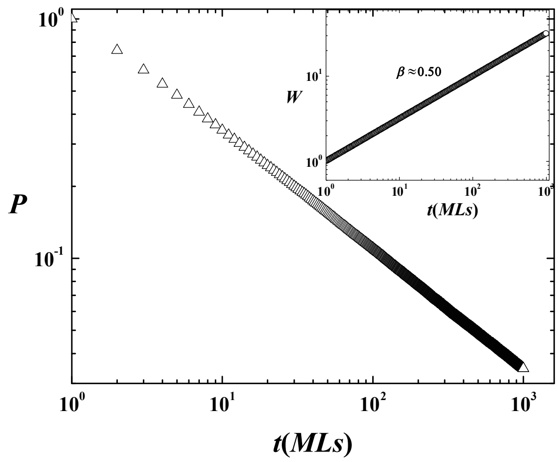

Figure 1 Persistence probability in the growth direction of the one dimensional RD model ($t_0 = 0$, $L = 100$ sites). Inset: interface width as a function of time.

In general, growth processes are not necessary up-down symmetric. The continuum equation in eq. (10) is linear so it cannot describe models without $h \rightarrow -h$ invariance. However, simulations still work for these models. An example of the up-down asymmetric model is the Das Sarma-Tamborenea (DT) model [23-24] whose diffusion rules allow the deposited atom to move along the surface to find the position that has at least one lateral bond. Our results show that the growth exponent of the DT model grown on one dimensional substrate is $\beta \approx 0.375$. The persistence probabilities of the model in the saturation regime are power-law decay as shown in Figure 2. The persistence exponents of the one dimensional DT model ($L = 100$ sites) are θ_+^S = 0.619 ± 0.001 and θ_-^S = 0.643 ± 0.001. For the DT model, the growth exponent is related to the positive persistence exponent which is approximately in accordant with the relation $\beta = \max[1-\theta_+^S, 1-\theta_-^S]$.

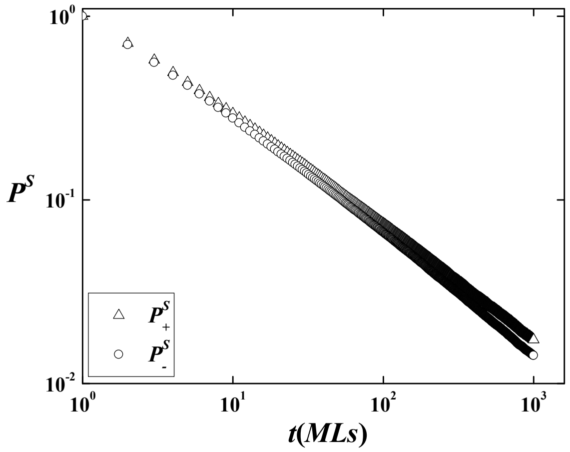

Figure 2 Positive and negative persistence probabilities of the one dimensional DT model ($t_0 = 10^6$ MLs, $L = 100$ sites).

Persistence probability in experiments

In experiments, the persistence concept is an important tool to study the stability of the crystalline structure. For example, the persistence probability helps predict the behavior of fluctuations at edges of crystal planes. In addition, for fabrication of nanoscale devices having two separating parts, one can use the persistence probability to estimate the time interval that two parts do not come into contact with each other due to fluctuations.

The persistence probability is also proved experimentally to have power-law behavior [12-13, 15]. Dougherty and coworkers [12] study fluctuations of Al steps on Si(111) vicinal surface at high temperature. The images of step positions are obtained by the line-scan scanning tunneling microscopy. The persistence probabilities for Al/Si(111) surface steps at 770 K, 870 K and 970 K are found to decay as a power-law with time. The averaged persistence exponent is θ = 0.77 ± 0.03 [12] which is consistent with the analytic value: θ = 3/4.

Conclusion

Understanding of persistence concept is helpful for characterizing dynamics of stochastic processes. All theoretical, simulation and experimental results shown above provide the conclusion that the persistence probability that the fluctuating variables never return to their initial value has power-law decay with time. The persistence exponent for a Markovian process can be calculated theoretically. For a non-Markovian process, the persistence exponent can be extracted by using some approximation, such as IIA or the perturbation theory. Moreover, the exponent is proved to be related to the growth exponent for the nonequilibrium interface growth processes. In addition, simulation and experiment help analyze dynamics of complicated stochastic systems.

Computer simulation and experimental studies of persistence confirm the analytic calculation of a process and give results that are in agreement with theory.

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