บทความวิจัย

คำตอบและเสถียรภาพของบางสมการ

ศุภโชค อิศริยาปัลลุย* และ วัชรพล พิมพ์เสริฐ

บทคัดย่อ

เรารำคำจำกัดความและเสถียรภาพของสมการเชิงฟังก์ชันของสมการที่คล้ายกับสมการเชิงฟังก์ชันเดวิลย์ ซึ่งก็คือ

\[ f(xy) + mf(x + y) = f(xy + x) + f(my) \]

และ

\[ f(xy) + mf(x + y) = f(xy + x) + mf(y) \]

เมื่อ \( x, y \in \mathbb{R} \) และ \( m \in \mathbb{R} \setminus \{0,1\} \) เป็นค่าคงที่ใดๆ

คำสำคัญ: สมการเชิงฟังก์ชัน สมการเชิงฟังก์ชันเดวิลย์ เสถียรภาพ

*ผู้นำพงษ์ประสานาน, e-mail: kalamung05@hotmail.com
Solutions and Stabilities of Some Equations

Supachoke Isariyapalakul* and Watcharapon Pimsert

ABSTRACT

We find the solutions and stabilities of two functional equations which are a generalized version of Davison functional equation, i.e.,

\[ f(xy) + mf(x + y) = f(xy + x) + f(my) \]

and

\[ f(xy) + mf(x + y) = f(xy + x) + mf(y) \]

where \( x, y \in \mathbb{R} \) and \( m \in \mathbb{R} \setminus \{0, 1\} \) is a constant.

Keywords: functional equation, Davison functional equation, stability

*Corresponding author, e-mail: kalamung05@hotmail.com
1. Introduction

In 1940, Ulam [1] introduced the following problem, which has since been referred to as a “stability” problem: let \( f \) be a mapping from a group \((G_1,+)\) to a metric group \((G_2,+)\) with metric \(d(\cdot,\cdot)\) such that

\[
d(f(x + y), f(x) + f(y)) \leq \varepsilon.
\]

Do there exist a group homomorphism \( L : G_1 \to G_2 \) and a constant \( \delta_\varepsilon > 0 \) such that \( d(f(x), L(x)) \leq \delta_\varepsilon \) for all \( x \in G_1 \)? This means that if we change a bit of the functional equation, then there is a little effect to its solution? In 1941, Hyers [2] proved that if \( f : E_1 \to E_2 \) is a function satisfying

\[
\| f(x + y) - f(x) - f(y) \| \leq \delta
\]

for all \( x, y \in E_1 \), where \( E_1 \) and \( E_2 \) are Banach spaces and \( \delta \) is a given positive number, then there exists a unique additive function \( T : E_1 \to E_2 \) such that

\[
\| f(x) - T(x) \| \leq \delta
\]

for all \( x \in E_1 \). If \( f \) is a real continuous function on \( \mathbb{R} \) satisfying

\[
| f(x + y) - f(x) - f(y) | \leq \delta,
\]

it was shown by Hyers and Ulam that there exists a constant \( k \) such that

\[
| f(x) - kx | \leq 2\delta.
\]


\[
f(xy) + f(x + y) = f(xy + x) + f(y)
\]

in the 17th ISFE (Oberwolfach). During the meeting, W. Benz presented that every continuous solution \( f : \mathbb{R} \to \mathbb{R} \) of (*) for all \( x, y \in \mathbb{R} \) is of the form \( f(x) = ax + b \) where \( a, b \) are real constants.

Next, in 1999, Jung and Sahoo [2] found the stability of (*) and its Pexider form:

\[
f(xy) + g(x + y) = h(xy + x) + k(y).
\]
In 2000, R. Girgensohn and K. Lajkó [4] solved the general solution of (*) and (**) for 
$x, y \in \mathbb{R}$ and for $x, y \in \mathbb{R}^+$, respectively.

In this paper, we propose the general solutions and stabilities of two functional equations which 
are an extended version of (*). Those are the functional equations

\[ f(xy) + mf(x + y) = f(xy + mx) + f(my) \] 

(1)

and

\[ f(xy) + mf(x + y) = f(xy + mx) + mf(y) \] 

(2)

where $x, y \in \mathbb{R}$ and $m \in \mathbb{R} \setminus \{0,1\}$.

2. Solutions

We find the solutions of (1) and (2), the results are

**Theorem 2.1** For a fixed $m \in \mathbb{R} \setminus \{0,1\}$. The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional 
equation (1) for all $x, y \in \mathbb{R}$ if and only if $f$ is an additive function such that $f(mx) = mf(x)$.

**Proof.** For a fixed $m \in \mathbb{R} \setminus \{0,1\}$. Suppose that $f$ is an additive function such that $f(mx) = mf(x)$.

Thus, we obtain

\[ f(x,y) + mf(x + y) = f(xy) + f(mx + my) \]

\[ = f(xy) + f(mx) + f(my) \]

\[ = f(xy + mx) + f(my), \]

for all $x, y \in \mathbb{R}$. So $f$ satisfies (1).

Next, we show the “if part” of this theorem. Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional 
equation (1) for all $x, y \in \mathbb{R}$.

Replacing $y$ by $y + m$ into (1), we get

\[ f(xy + mx) + mf(x + y + m) = f(xy + 2mx) + f(my + m^2). \] 

(2.1)

By adding (1) and (2.1), we have

\[ f(xy) + mf(x + y) + mf(x + y + m) = f(my) + f(xy + 2mx) + f(my + m^2). \]

From the above equation, we substitute $x$ by $\frac{x}{2}$ and $y$ by $2y$:

\[ f(xy) + mf(\frac{x}{2} + 2y) + mf(\frac{x}{2} + 2y + m) = f(2my) + f(xy + mx) + f(2my + m^2). \] 

(2.2)

Next, we subtract (1) from (2.2), that is, (2.2) – (1), to get
\[ mf\left(\frac{x}{2} + 2y\right) + mf\left(\frac{x}{2} + 2y + m\right) - mf(x + y) = f(2my) + f(2my + m^2) - f(my). \]

Then, replacing \( x \) by \( x - y \) in the above equation, we get

\[ mf\left(\frac{x}{2} + \frac{3y}{2}\right) + mf\left(\frac{x}{2} + \frac{3y}{2} + m\right) - mf(x) = f(2my) + f(2my + m^2) - f(my). \]

Replacing \( y \) by \( \frac{y}{3} \) in the last equation:

\[ mf\left(\frac{x}{2} + \frac{y}{2}\right) + mf\left(\frac{x}{2} + \frac{y}{2} + m\right) - mf(x) = f\left(\frac{2my}{3}\right) + f\left(\frac{2my}{3} + m^2\right) - f\left(\frac{my}{3}\right). \]

Now, it is of the pexider form:

\[ A_3(x + y) = A_2(x) + A_1(y) \]

where

\[ A_1(t) := f\left(\frac{2mt}{3}\right) + f\left(\frac{2mt}{3} + m^2\right) - f\left(\frac{mt}{3}\right), \tag{2.3} \]

\[ A_2(t) := mf(t), \tag{2.4} \]

\[ A_3 := mf\left(\frac{t}{2}\right) + mf\left(\frac{t}{2} + m\right). \tag{2.5} \]

So there exists an additive function \( A : \mathbb{R} \to \mathbb{R} \) and constants \( d_1, d_2 \in \mathbb{R} \) such that

\[ A_1(x) = A(x) + d_1, \tag{2.6} \]

\[ A_2(x) = A(x) + d_2 \text{, and} \tag{2.7} \]

\[ A_3(x) = A(x) + d_1 + d_2. \tag{2.8} \]

From (2.4) and (2.7), we get

\[ f(x) = \frac{1}{m} A_2(x) = \frac{1}{m} A(x) + \frac{d_2}{m}. \tag{2.9} \]

From (2.3) and (2.6), we have

\[ A(x) + d_1 = f\left(\frac{2mx}{3}\right) + f\left(\frac{2mx}{3} + m^2\right) - f\left(\frac{mx}{3}\right). \tag{2.10} \]
Hence, from (2.9) and (2.10), we obtain that
\[
A(x) + d_1 = \frac{1}{m} \left( A\left( \frac{2mx}{3} \right) + \frac{d_2}{m} + \frac{1}{m} A\left( \frac{2mx}{3} + m^2 \right) + d_2 - \frac{1}{m} A\left( \frac{mx}{3} \right) - \frac{d_2}{m} \right)
\]
\[
= \frac{1}{m} A(mx) + \frac{1}{m} A(m^2) + \frac{d_2}{m}.
\] (2.11)

From (2.5), (2.8) and (2.9), we get
\[
A(x) + d_1 + d_2 = mf\left( \frac{x}{2} \right) + mf\left( \frac{x}{2} + m \right)
\]
\[
A(x) + d_1 + d_2 = A\left( \frac{x}{2} \right) + d_2 + A\left( \frac{x}{2} + m \right) + d_2
\]
\[
A(x) + d_1 = A(x) + A(m) + d_2
\]
\[
d_1 = A(m) + d_2.
\] (2.12)

Next, we substitute \(x\) by \(-m\) in (2.11):
\[
A(-m) + d_1 = \frac{1}{m} A(-m^2) + \frac{1}{m} A(m^2) + \frac{d_2}{m}
\]
\[
mA(-m) + md_1 = d_2
\]
\[
m(-A(m) + d_1) = d_2.
\]

By (2.12), we obtain
\[
md_2 = d_2
\]
\[
(m-1)d_2 = 0.
\]

Since \(m \neq 1\), so
\[
d_2 = 0.
\] (2.13)

And replacing \(x\) by 0 in (2.11), we have
From (2.12) and (2.13), so

$$mA(m) = A(m^2).$$

(2.14)

By using (2.11), (2.12), (2.13) and (2.14), we obtain that

$$A(x) + d_1 = \frac{1}{m} A(mx) + \frac{1}{m} A(m^2) + \frac{d_2}{m}$$

$$mA(x) + md_1 = A(mx) + A(m^2)$$

$$mA(x) + mA(m) = A(mx) + mA(m)$$

$$mA(x) = A(mx).$$

So, by (2.9), we get

$$f(x) = \frac{1}{m} A(x)$$

where \( A: \mathbb{R} \rightarrow \mathbb{R} \) is an additive function such that

$$f(mx) = \frac{1}{m} A(mx) = \frac{1}{m} mA(x) = A(x) = mf(x),$$

i.e., \( f \) is an additive function where \( f(mx) = mf(x) \).

**Theorem 2.2** For a fixed \( m \in \mathbb{R} \setminus \{0, 1\} \). If the function \( f: \mathbb{R} \rightarrow \mathbb{R} \) satisfies the functional equation (2) for all \( x, y \in \mathbb{R} \), then \( f \) is of the form

$$f(x) = A(x) + b$$

where \( A: \mathbb{R} \rightarrow \mathbb{R} \) is an additive function and \( b \in \mathbb{R} \) is a constant.

**Proof.** Suppose that \( f: \mathbb{R} \rightarrow \mathbb{R} \) satisfies the functional equation (2) for all \( x, y \in \mathbb{R} \). First, we substitute \( y \) by \( y + m \) in (2):

$$f(xy + mx) + mf(x + y + m) = f(xy + 2mx) + mf(y + m).$$

(2.15)

By adding (2) and (2.15), we have

$$f(xy) + mf(x + y) + mf(x + y + m) = mf(y) + f(xy + 2mx) + mf(y + m).$$

Replacing \( x \) by \( \frac{x}{2} \) and \( y \) by \( 2y \) in the last equation, we get

$$f(xy) + mf(\frac{x}{2} + 2y) + mf(\frac{x}{2} + 2y + m) = mf(2y) + f(xy + mx) + mf(2y + m).$$

(2.16)

From (2) and (2.16), we get
\[ mf\left(\frac{x}{2} + 2y\right) + mf\left(\frac{x}{2} + 2y + m\right) - mf(x + y) = mf(2y) + mf(2y + m) - mf(y) \]
\[ f\left(\frac{x}{2} + 2y\right) + f\left(\frac{x}{2} + 2y + m\right) - f(x + y) = f(2y) + f(2y + m) - f(y) \]

Next, replacing \( x \) by \( x - y \):

\[ f\left(\frac{x}{2} + \frac{3y}{2}\right) + f\left(\frac{x}{2} + \frac{3y}{2} + m\right) - f(x) = f(2y) + f(2y + m) - f(y) \]

By substituting \( y \) by \( \frac{y}{3} \) in the above equation, we obtain

\[ f\left(\frac{x}{2} + \frac{y}{2}\right) + f\left(\frac{x}{2} + \frac{y}{2} + m\right) = f(x) + f\left(\frac{2y}{3}\right) + f\left(\frac{2y}{3} + m\right) - f\left(\frac{y}{3}\right) \]

We see that the above equation is of the pexider form

\[ A_3(x + y) = A_2(x) + A_1(y) \]

where

\[ A_1(t) := f\left(\frac{2t}{3}\right) + f\left(\frac{2t}{3} + m\right) - f\left(\frac{t}{3}\right) \]
\[ A_2(t) := f(t) \]
\[ A_3(t) := f\left(\frac{t}{2}\right) + f\left(\frac{t}{2} + m\right) \]

So the general solutions are

\[ A_1(t) := A(t) + a \]
\[ A_2(t) := A(t) + b \]
\[ A_3(t) := A(t) + a + b \]

where \( A: \mathbb{R} \rightarrow \mathbb{R} \) is an additive function and \( a, b \) are constants. So

\[ f(x) = A(x) + b \]

3. Stabilities

In this section, we consider the stabilities of (1) and (2) with new condition, i.e., \( m \) is a nonzero real number. The results are stated below.

**Theorem 3.1** For a fixed \( m, \delta \in \mathbb{R} \) with \( m \neq 0 \) and \( \delta > 0 \). If the function \( f: \mathbb{R} \rightarrow \mathbb{R} \) satisfies the inequality
for all $x, y \in \mathbb{R}$, then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - f(0) - \frac{1}{m} A(x)| \leq \frac{12\delta}{|m|}$$

for all $x \in \mathbb{R}$.

Proof. First, we replace $y$ by $y + m$ in (3.1):

$$|f(xy + mx) + mf(x + y + m) - f(xy + 2mx) - f(my + m^2)| \leq \delta. \quad (3.2)$$

By adding (3.1) and (3.2), we get

$$|f(xy) + mf(x + y) - f(my) + mf(x + y + m) - f(xy + 2mx) - f(my + m^2)|$$
$$\leq |f(xy) + mf(x + y) - f(xy + mx) - f(my)| + |f(xy + mx) + mf(x + y + m) - f(xy + 2mx) - f(my + m^2)| \leq 2\delta.$$

Replacing $x$ by $\frac{x}{2}$ and $y$ by $2y$ in the above inequality, we obtain

$$|f(xy) + mf\left(\frac{x}{2} + 2y\right) - f(2my) + mf\left(\frac{x}{2} + 2y + m\right)$$
$$- f(xy + mx) - f(2my + m^2)| \leq 2\delta. \quad (3.3)$$

From (3.1) and (3.3), we get

$$|mf\left(\frac{x}{2} + 2y\right) - f(2my) + mf\left(\frac{x}{2} + 2y + m\right) - f(2my + m^2) - mf(x + y) + f(my)|$$
$$\leq |f(xy) - mf(x + y + m) + f(xy + mx) + f(my)|$$
$$+ |f(xy) + mf\left(\frac{x}{2} + 2y\right) - f(2my) + mf\left(\frac{x}{2} + 2y + m\right) - f(xy + mx) - f(2my + m^2)|$$
$$\leq 3\delta.$$

Replacing $x$ by $x - y$ in the above inequality, we have

$$|mf\left(\frac{x}{2} + \frac{3y}{2}\right) + mf\left(\frac{x}{2} + \frac{3y}{2} + m\right) - mf(x) - f(2my) - f(2my + m^2) + f(my)| \leq 3\delta.$$
Letting $y$ by $\frac{y}{3}$, we obtain

$$\left| mf\left(\frac{x}{2} + \frac{y}{2}\right) + mf\left(\frac{x}{2} + \frac{y}{2} + m\right) - mf(x) - f\left(\frac{2my}{3}\right) - f\left(\frac{2my}{3} + m^2\right) + f\left(\frac{my}{3}\right)\right| \leq 3\delta . \quad (3.4)$$

Next, we let $g, h : \mathbb{R} \rightarrow \mathbb{R}$ define by

$$g(x) = \frac{f\left(\frac{2mx}{3}\right) + f\left(\frac{2mx}{3} + m\right) - f\left(\frac{mx}{3}\right)}{2}$$
$$h(x) = mf\left(\frac{x}{2}\right) + mf\left(\frac{x}{2} + m\right). \quad (3.5)$$

From (3.4) and (3.5), we get

$$| h(x + y) - mf(x) - g(y) | \leq 3\delta . \quad (3.6)$$

Replacing $y$ by 0 in (3.6), we obtain

$$| h(x) - mf(x) - g(0) | \leq 3\delta . \quad (3.7)$$

Similarly, we replace $x$ by 0 in (3.6), we have that

$$| h(y) - mf(0) - g(y) | \leq 3\delta . \quad (3.8)$$

Next, we define

$$H(x) = h(x) - mf(0) - g(0). \quad (3.9)$$

By using (3.6), (3.7), (3.8) and (3.9), we have

$$| H(x + y) - H(x) - H(y) | = | h(x + y) - h(x) - h(y) + mf(0) + g(0) |$$
$$\leq | h(x + y) - mf(x) - g(y) | + | mf(x) - h(x) + g(0) | + | g(y) - h(y) + mf(0) |$$
$$\leq 9\delta . \quad (3.10)$$

Now using Hyers theorem [2], we get that

$$| H(x) - A(x) | \leq 9\delta , \quad (3.11)$$

where $A : \mathbb{R} \rightarrow \mathbb{R}$ is a unbounded additive function such that $A(x) = \lim_{n \to \infty} \frac{H\left(2^n x\right)}{2^n}$. By (3.7), (3.9) and (3.10), we have

$$| mf(x) - mf(0) - A(x) | \leq | -h(x) + mf(x) + g(0) | + | h(x) - mf(0) - g(0) - A(x) |$$
$$\leq 12\delta .$$
Thus, we obtain

\[ |f(x) - f(0) - \frac{1}{m} A(x)| \leq \frac{12\delta}{|m|} \]

for all \( x, y \in \mathbb{R} \).

**Theorem 3.2** For a fixed \( m, \delta \in \mathbb{R} \) with \( m \neq 0 \) and \( \delta > 0 \). If the function \( f: \mathbb{R} \to \mathbb{R} \) satisfies the inequality

\[ |f(xy) + mf(x + y) - f(xy + mx) - mf(y)| \leq \delta, \tag{3.12} \]

then there exists a unique additive function \( A: \mathbb{R} \to \mathbb{R} \) such that

\[ |f(x) - f(0) - A(x)| \leq \frac{12\delta}{|m|} \]

for all \( m \in \mathbb{R} \).

**Proof.** First, we substitute \( y \) by \( y + m \) in (3.12):

\[ |f(xy + mx) + mf(x + y + m) - f(xy + 2mx) - mf(y + m)| \leq \delta. \tag{3.13} \]

By adding (3.12) and (3.13), we get

\[
|f(xy) + mf(x + y) - mf(y) + mf(x + y + m) - f(xy + 2mx) - mf(y + m)| \\
\leq |f(xy) + mf(x + y) - f(xy + mx) - mf(y)| \\
+ |f(xy + mx) + mf(x + y + m) - f(xy + 2mx) - mf(y + m)| \\
\leq 2\delta.
\]

Replacing \( x \) by \( \frac{x}{2} \) and \( y \) by \( 2y \) in the above inequality, we get

\[ |f(xy) + mf(\frac{x}{2} + 2y) - mf(2y) + mf(\frac{x}{2} + 2y + m) - f(xy + mx) - mf(2y + m)| \leq 2\delta. \tag{3.14} \]
From (3.12) and (3.14), we have that

\[
|mf\left(\frac{x}{2} + 2y\right) - mf(2y) + mf\left(\frac{x}{2} + 2y + m\right) - mf(x + y) + mf(y)|
\]

\[
\leq | -f(xy) - mf(x + y) + f(xy + mx) + mf(y)|
\]

\[
+ | f(xy) + mf\left(\frac{x}{2} + 2y\right) - mf(2y) + mf\left(\frac{x}{2} + 2y + m\right) - f(xy + mx) - mf(2y + m)|
\]

\[3\delta.\]

So we get

\[
|f\left(\frac{x}{2} + 2y\right) + f\left(\frac{x}{2} + 2y + m\right) - f(x + y) - f(2y) - f(2y + m) + f(y)| \leq \frac{3\delta}{|m|}.
\]

From the last inequality, we replace \(x\) by \(x - y\) to get

\[
|f\left(\frac{x}{2} + \frac{3y}{2}\right) + f\left(\frac{x}{2} + \frac{3y}{2} + m\right) - f(x) - f(2y) - f(2y + m) + f(y)| \leq \frac{3\delta}{|m|}.
\]

And then substitute \(y\) by \(\frac{y}{3}\):

\[
\left|f\left(\frac{x}{2} + \frac{y}{2}\right) + f\left(\frac{x}{2} + \frac{y}{2} + m\right) - f(x) - f\left(\frac{2y}{3}\right) - f\left(\frac{2y}{3} + m\right) + f\left(\frac{y}{3}\right)\right| \leq \frac{3\delta}{|m|}. \tag{3.15}
\]

Next, we define the function \(g, h : \mathbb{R} \rightarrow \mathbb{R}\) by

\[
g(x) = \left\{ f\left(\frac{2x}{3}\right) + f\left(\frac{2x}{3} + m\right) - f\left(\frac{x}{3}\right) \right\}
\]

\[
h(x) = f\left(\frac{x}{2}\right) + f\left(\frac{x}{2} + m\right). \tag{3.16}
\]

From (3.15) and (3.16), we get

\[
|h(x + y) - f(x) - g(y)| \leq \frac{3\delta}{|m|}. \tag{3.17}
\]

From (3.17), we substitute \(y\) by 0 and \(x\) by 0 respectively to get

\[
|h(x) - f(x) - g(0)| \leq \frac{3\delta}{|m|}. \tag{3.18}
\]
and

\[
| h(y) - f(0) - g(y) | \leq \frac{3\delta}{m}.
\]  

(3.19)

Next, we define

\[
H(x) = h(x) - f(0) - g(0)
\]

(3.20)

Using (3.17), (3.18), (3.19) and (3.20), we obtain

\[
\begin{align*}
| H(x + y) - H(x) - H(y) | &= | h(x + y) - h(x) - h(y) + f(0) + g(0) | \\
&\leq | h(x + y) - f(x) - g(y) | + | f(x) - h(x) + g(0) | + | g(y) - h(y) + f(0) | \\
&\leq \frac{9\delta}{m}.
\end{align*}
\]

By Hyers theorem [2], we obtain

\[
| H(x) - A(x) | \leq \frac{9\delta}{m}
\]  

(3.21)

where \( A : \mathbb{R} \to \mathbb{R} \) is a unique additive function such that \( A(x) = \lim_{n \to \infty} \frac{H(2^n x)}{2^n} \). Now using (3.18), (3.20) and (3.21), we obtain

\[
\begin{align*}
| f(x) - f(0) - A(x) | &\leq | f(x) + g(0) - h(x) | + | h(x) - f(0) - g(0) - A(x) | \\
&\leq | h(x) - f(x) - g(0) | + | H(x) - A(x) | \\
&\leq \frac{12\delta}{m}
\end{align*}
\]

for all \( x \in \mathbb{R} \).

From Theorem 3.1 and Theorem 3.2, if we let \( m = 1 \), then we obtain the result of Jung and Sahoo [2]:

**Corollary 3.3** If the function \( f : \mathbb{R} \to \mathbb{R} \) satisfies the inequality

\[
| f(xy) + f(x + y) - f(xy + x) - f(y) | \leq \delta
\]

for all \( x, y \in \mathbb{R} \), then there exists a unique additive function \( A : \mathbb{R} \to \mathbb{R} \) such that

\[
| f(x) - f(0) - A(x) | \leq 12\delta
\]

for all \( x \in \mathbb{R} \).
References


ได้รับบทความวันที่ 15 พฤศจิกายน 2556
ยื่นรับพิมพ์วันที่ 24 ธันวาคม 2556